Chapter 3: The Derivative
3.1: Limits

- Algebra deals with static situations (what will something cost in 3 years?)
- Calculus deals with dynamic situations (when does the growth of a population begin to slow down)
- The concept of a limit is a fundamental concept in calculus. One use is in applications that include maximizations.
- The idea of the limit of a function is what connects algebra and geometry to the mathematics of calculus.

There are multiple methods to Finding Limits: Case 1

- If a polynomial function is defined at a point, then the limit is obtained by direct substitution

- Example: $\lim_{x\to 2} 3x^2 - 2x + 27$

  $= 3(2)^2 - 2(2) + 27 = 35$

- You Do: $\lim_{x\to -2} x^3 - 3x + 12$ 10

Case 2

- In a rational function, if direct substitution results in a 0 in both the numerator and denominator, (this is called the indeterminant form) factor both, reduce to lowest terms, then use direct substitution.

- Example: $\lim_{x \to 3} \frac{2x^2 - 6x}{x - 3} = \frac{0}{0}, so \frac{2x(x-3)}{x-3} = 2x = 2(3) = 6$

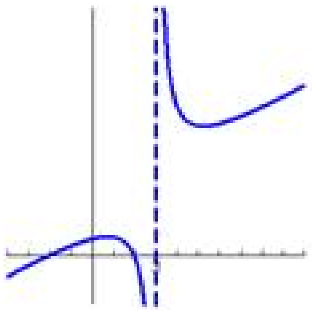
Case 3

- In a rational function, if direct substitution results in division by zero, evaluate small increments close to a, from both the left and the right. Note that $x = a$ is a vertical asymptote. In order for a function to have a limit at a point, its right- and left-hand limits have to be the same. If not, the limit does not exist. The limit may occur at the vertical asymptote.

- Example $\lim_{x\to 3}\dfrac{x^2-4}{x-3}=\dfrac{5}{0}$

Set up a table for x getting closer to 3 from the left and the right

x	f(x)
3.1	56.1
3.01	506.01
3.001	5006.01
2.9	-44.1
2.99	-494.0
2.999	-4994.01



As x gets closer to 3 from the left, the y-coordinates get forever smaller ($-\infty$). As x gets closer to 3 from the right, f(x) increases without bound (∞). Since $\lim_{x\to 3-}\dfrac{x^2-4}{x-3}=-\infty$ and $\lim_{x\to 3+}\dfrac{x^2-4}{x-3}=\infty$, the limit does not exist.

Example $\lim_{x \to 2} \frac{x+2}{x-2}$

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The limit as $x \to 2^{-} = -\infty$

The limit as $x \to 2^{+} = \infty$

Therefore, the limit DNE

Definition of a Limit

- If f(x) becomes arbitrarily close to a unique number L as x approaches c from either side, the limit of f(x) as x approaches c is L. This is written $\lim_{x \to c} f(x) = L$

Case 4

- You can estimate the limit by substituting increasingly larger values for n. (look at 100, 1000, 100000000…)
- Ex: Find $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right)$. When n = 100, $\sin\left(\frac{1}{100}\right) \approx 0.01$
- As n gets larger and larger, $\frac{1}{n}$ approaches

  0 and so does $\sin\left(\frac{1}{n}\right)$. So, $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$.

Case 4 Continued

- Ex: Find $\lim_{n\to\infty}(0.99)^n$:

  $(0.99)^{1000} = 4.3 \text{ x } 10^{-5}.$

  $(0.99)^{10,000} = 2.2 \text{ x } 10^{-44}$

- So, $\lim_{n\to\infty}(0.99)^n = 0$

Case 5

- Divide the numerator and denominator by the highest power of n that occurs in the *denominator*.

Example

$$\text{Find } \lim_{n\to\infty} \frac{n^2+1}{2n^2-3n}$$

What is the highest power of n in the denominator?

$$\frac{\frac{\cancel{n^2}}{\cancel{n^2}}+\frac{1}{n^2}}{\frac{2\cancel{n^2}}{\cancel{n^2}}-\frac{3n}{n^2}} = \frac{1+0}{2-0} = \frac{1}{2}$$

You Do

$$Find\ \lim_{n\to\infty}\frac{5n^2+\sqrt{n}}{3n^3+7}$$

$$\lim_{n\to\infty}\frac{5n^2+\sqrt{n}}{3n^3+7}=0$$

Increasing Without Bound

- Sometimes, the terms of a sequence increase without bound, in which case, the limit does not exist.

- Ex: 3, 7, 11, 15, … $4n - 1$, … so
- Ex: -10, -100, -1000, … -10n, … so

You Do

$$\text{Show that the } \lim_{n\to\infty} \frac{7n^3}{n^2-5} = \infty$$

$$\frac{\frac{7n^3}{n^2}}{\frac{n^2}{n^2}-\frac{5}{n^2}} = \frac{7n}{1} = 7n = \infty$$

Therefore, the limit DNE