Chapter 4
Calculating the Derivative

JMerrill, 2009

Review

- Find the derivative of $(3x - 2x^2)(5 + 4x)$
- $-24x^2 + 4x + 15$
- Find the derivative of $\dfrac{5x-2}{x^2+1}$
- $$\frac{-5x^2+4x+5}{(x^2+1)^2}$$

4.3
The Chain Rule

Composition of Functions

- A composition of functions is simply putting 2 functions together—one inside the other.
- Example: In order to convert Fahrenheit to Kelvin we have to use a 2-step process by first converting Fahrenheit to Celsius.

$$C = (F - 32)\frac{5}{9}$$

- $89^{o}F = 31.7^{o}C$
- $31.7^{o}C = 304.7K$

$$K = C + 273$$

- But if we put 1 function inside the other function, then it is a 1-step process.

Composition of Functions

The composite of f(x) and g(x) is denoted $(f \circ g)(x)$ which means the same as f(g(x)).

- We are used to writing f(x). f(g(x)) simply means that g(x) is our new x in the f equation.
- We can also go the other way. $(g \circ f)(x)$ means g(f(x)).

Given $f(x) = 4x^2 - 2x \quad g(x) = 2x$

$f(g(3)) =$

$g(3) = 6$

$= f(6)$

$= 4(6)^2 - 2(6)$

$= 144 - 12$

$\boxed{= 132}$

Given $f(x) = \frac{1}{x}$ $g(x) = x + 1$

$$(f \circ g)(x) =$$

$$= f(g(x))$$

$$= f(x+1)$$

$$\boxed{= \frac{1}{x+1}}$$

g(x) = x+1

Substitute x+1
In place of the
x in the f equation

$$(g \circ f)(x) =$$

$$= g(f(x))$$

$$= g\left(\frac{1}{x}\right)$$

$$\boxed{= \frac{1}{x} + 1}$$

=

The new x in the g equation

The Chain Rule

CHAIN RULE

If y is a function of u, say $y = f(u)$, and if u is a function of x, say $u = g(x)$, then $y = f(u) = f[g(x)]$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Chain Rule Example

- Use the chain rule to find $D_x(x^2 + 5x)^8$
- Let $u = x^2 + 5x$
- Let $y = u^8$

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

$$= 8u^7(2x+5)$$

$$= 8(x^2+5x)^7(2x+5)$$

Another way to think of it: The derivative of the outside times the derivative of the inside

Chain Rule – You Try

- Use the chain rule to find $D_x(3x - 2x^2)^3$
- Let $u = 3x - 2x^2$
- Let $y = u^3$

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

The derivative of the outside times the derivative of the inside

$$= 3u^2(3-4x)$$

$$= 3\left(3x-2x^2\right)^2(3-4x)$$

Chain Rule

- Find the derivative of $y = 4x(3x + 5)^5$
- This is the Product Rule inside the Chain Rule.
- Let $u = 3x + 5$; $y = u^5$

$$4x\left[5u^4(3)\right] + (3x+5)^5(4)$$

$$4x\left[5(3x+5)^4(3)\right] + 4(3x+5)^5$$

$$4x\left(15(3x+5)^4\right) + 4(3x+5)^5$$

$$60x(3x+5)^4 + 4(3x+5)^5$$

Chain Rule

$$= 60x(3x+5)^4 + 4(3x+5)^5$$

Factor out the common factor

$$= 4(3x+5)^4\left[15x + (3x+5)\right]$$

$$= 4(3x+5)^4(18x+5)$$

Chain Rule

- Find the derivative of $\frac{(3x+2)^7}{x-1}$
- This is the Quotient Rule in the Chain Rule
- Let u = 3x + 2; let y = u^7

$$= \frac{(x-1)\left[7u^6(3)\right]-(3x+2)^7(1)}{(x-1)^2}$$

$$= \frac{(x-1)\left[7(3x+2)^6(3)\right]-(3x+2)^7}{(x-1)^2}$$

$$= \frac{21\left[(x-1)(3x+2)^6\right]-(3x+2)^7}{(x-1)^2}$$

Chain Rule

$$= \frac{21\left[(x-1)(3x+2)^6\right]-(3x+2)^7}{(x-1)^2}$$

Factor out the common factor

$$= \frac{(3x+2)^6\left[21(x-1)-(3x+2)\right]}{(x-1)^2}$$

$$= \frac{(3x+2)^6\left[21x-21-3x-2\right]}{(x-1)^2}$$

$$= \frac{(3x+2)^6(18x-23)}{(x-1)^2}$$

4.4
Derivatives of Exponential Functions

Derivative of e^x

DERIVATIVE OF e^x

$$\frac{d}{dx}(e^x) = e^x$$

Derivative of a^x

DERIVATIVE OF a^x

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

(The derivative of an exponential function is the original function times the natural logarithm of the base.)

$$D_x 3^x = (\ln 3)3^x$$

Other Derivatives

DERIVATIVE OF $a^{g(x)}$ AND $e^{g(x)}$

$$\frac{d}{dx}\left(a^{g(x)}\right) = (\ln a)a^{g(x)}g'(x)$$

and

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g'(x)$$

Examples – Find the Derivative

- $y = e^{5x}$

$$= e^{g(x)}(g'(x))$$

$$= e^{5x}(5) = 5e^{5x}$$

Examples – Find the Derivative

- $y = 3^{2x+1}$

$$= \ln a\left(a^{g(x)}\right)g'(x)$$

$$= \ln 3\left(3^{2x+1}\right)(2)$$

$$= 2\ln 3\left(3^{2x+1}\right)$$

Example

- Find $\frac{dy}{dx}$ if $y = e^{x^2+1}\sqrt{5x+2}$
- Use the product rule

$$y = e^{x^2+1}\left(D_x\left(5x+2\right)^{\frac{1}{2}}\right) + \sqrt{5x+2}\left(D_x\left(e^{x^2+1}\right)\right)$$

$$= \frac{1}{2}\left(5x+2\right)^{\frac{-1}{2}}(5)$$

$$= e^{x^2+1}(2x)$$

$$= \frac{5}{2\sqrt{5x+2}}$$

Example

$$y = e^{x^2+1}\left(D_x\left(5x+2\right)^{\frac{1}{2}}\right) + \sqrt{5x+2}\left(D_x\left(e^{x^2+1}\right)\right)$$

$$= e^{x^2+1}\frac{5}{2\sqrt{5x+2}} + \sqrt{5x+2}\left(2xe^{x^2+1}\right)$$

$$= \frac{5e^{x^2+1}}{2\sqrt{5x+2}} + \sqrt{5x+2}\left(2xe^{x^2+1}\right)$$

Example Continued

$$= \frac{5e^{x^2+1}}{2\sqrt{5x+2}} + \left(\left(2xe^{x^2+1}\right)\sqrt{5x+2}\right)\left(\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}}\right)$$

$$= \frac{5e^{x^2+1} + e^{x^2+1}(4x)(5x+2)}{2\sqrt{5x+2}}$$

Get a common denominator to add the 2 parts together

$$= \frac{e^{x^2+1}\left(5 + 4x(5x+2)\right)}{2\sqrt{5x+2}}$$

$$= \frac{e^{x^2+1}\left(20x^2 + 8x + 5\right)}{2\sqrt{5x+2}}$$

4.5
Derivatives of Logarithmic Functions

Definition

DERIVATIVE OF $\log_a x$

$$\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$$

(The derivative of a logarithmic function is the reciprocal of the product of the variable and the natural logarithm of the base.)

Bases – a side note

- Everything we do is in Base 10.
  - We count up to 9, then start over. We change our numbering every 10 units.

1
2
3
4
5
6
7
8
9
10

Ones Place

11
12
13
14
15
16
17
18
19
20

One group of ten and 1, 2, 3…ones

21
22
23…

Two tens and …ones

Bases

- The Yuki of Northern California used Base 8.
  - They counted up to 7, then started over. The numbering changed every 8 units.

1
2
3
4
5
6
7
10
11
12

Ones
Place

13
14
15
16
17
20
21
22
23
24

One
eight
and
3…ones

25
26
27…

Two
eights
and
…one
s

So, 17 in Base 8 = 15 in Base 10

25_8 = 2 eights + 5 ones = 21

Bases

- The Mayans used Base 20.
- The Sumerians and people of Mesopotamia used Base 60.

Definition

DERIVATIVE OF ln x

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Example

- Find f'(x) if f(x) = ln 6x
- Remember the properties of logs
- ln 6x = ln 6 + ln x

$$\frac{d}{dx}(\ln 6) + \frac{d}{dx}(\ln x)$$

$$= 0 + \frac{1}{x} = \frac{1}{x}$$

Definitions

DERIVATIVE OF $\log_a|x|$, $\log_a|g(x)|$, $\ln|x|$, AND $\ln|g(x)|$

$$\frac{d}{dx}[\log_a|x|] = \frac{1}{(\ln a)x} \qquad \frac{d}{dx}[\log_a|g(x)|] = \frac{1}{\ln a}\cdot\frac{g'(x)}{g(x)}$$

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \qquad \frac{d}{dx}[\ln|g(x)|] = \frac{g'(x)}{g(x)}$$

Examples – Find the Derivatives

- y = ln 5x
- If g(x) = 5x, then g'(x) = 5

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{5}{5x} = \frac{1}{x}$$

F'(x)

- f(x) = 3x ln x^2
- Product Rule

$$f'(x) = (3x)\left[\frac{d}{dx}\ln x^2\right] + \left(\ln x^2\right)(3)$$

$$= 3x\left(\frac{2x}{x^2}\right) + \left(\ln x^2\right)(3)$$

$$= 6 + 3\ln x^2$$