6.4
Implicit Differentiation

JMerrill, 2009

Help

- Paul's Online Math Notes
- YouTube Tutorial
- Visual Calculus
- Google Video Tutorial
- MindBites Video Tutorial
- Visual Calculus Practice Problems

Explicit/Implicit

- Explicit functions:

$$y = 3x - 2$$

$$y = x^2 + 5$$

- Implicit functions:

$$y^2 + 2yx\ 4x^2 = 0$$

$$y^5 - 3y^2x^2 + 2 = 0$$

Why Implicit Differentiation?

- When an applied problem involves an equation not in explicit form, implicit differentiation is used to locate extrema or to find rates of change.

Process for Implicit Differentiation

- To find dy/dx
- Differentiate both sides with respect to x (y is assumed to be a function of x, so d/dx)
- Collect like terms (all dy/dx on the same side, everything else on the other side)
- Factor out the dy/dx and solve for dy/dx

Example

- Find dy/dx if $3xy + 4y^2 = 10$
- Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(3xy + 4y^2\right) = \frac{d}{dx}10$$

- Use the product rule for (3x)(y)
- (The derivative of y is dy/dx)

$$(3x)\frac{dy}{dx} + y(3) = (3x)\frac{dy}{dx} + 3y$$

Example Con't

$$\frac{d}{dx}\left(3xy + 4y^2\right) = \frac{d}{dx}10$$

Since y is assumed to be some function of x, use the chain rule for $4y^2$

$$\frac{d}{dx}\left(4y^2\right) = 4\left(2y^1\right)\frac{dy}{dx} = 8y\frac{dy}{dx}$$

Example Con't

$$\frac{d}{dx}\left(3xy+4y^2\right)=\frac{d}{dx}10$$

$$(3x)\frac{dy}{dx}+3y+8y\frac{dy}{dx}=0$$

Combine like terms

$$(3x)\frac{dy}{dx}+8y\frac{dy}{dx}=-3y$$

Example Con't

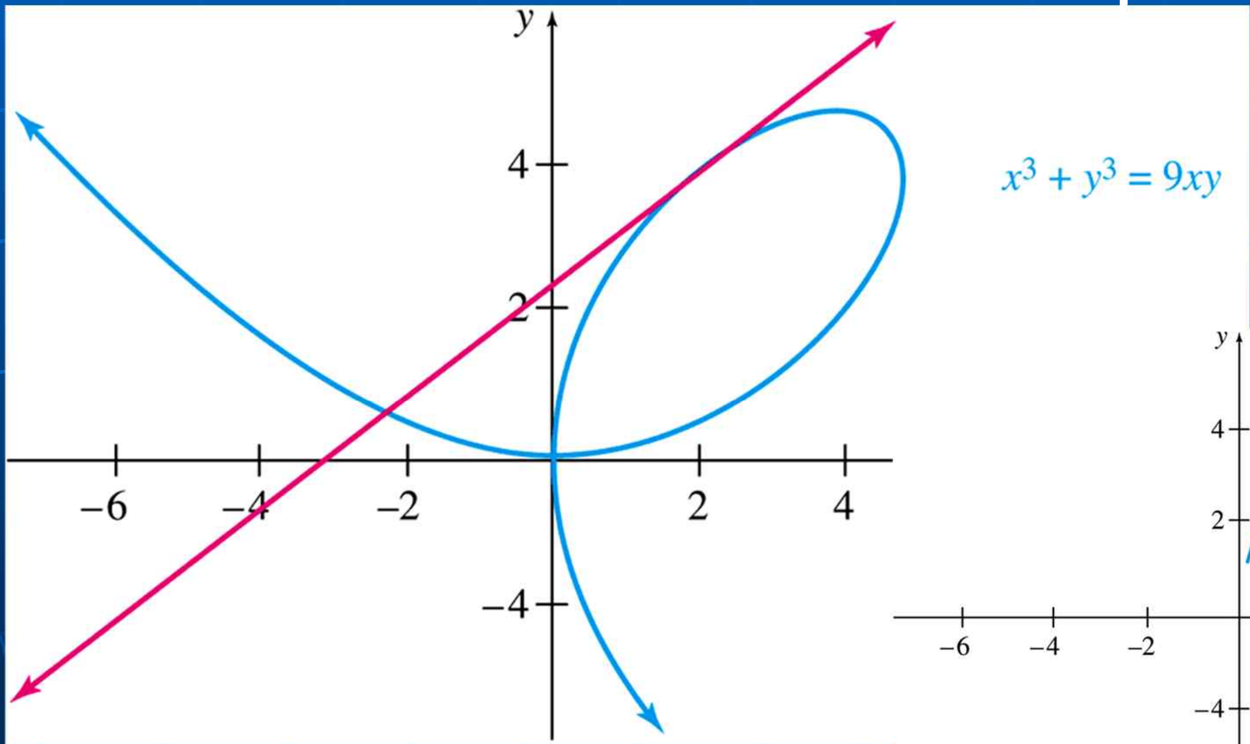
Factor and solve

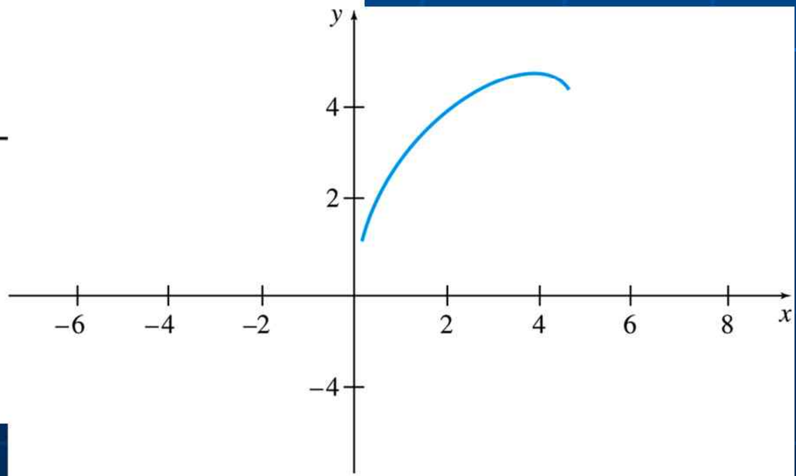
$$\frac{dy}{dx}(3x+8y) = -3y$$

$$\frac{dy}{dx} = \frac{-3y}{(3x+8y)}$$

Tangent Lines

- Find the equation of the tangent line to the curve below at the point (2,4)





Tangent Example

- $x^3 + y^3 = 9xy$

Now what?

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y$$

$$3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 9x) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3(3y - x^2)}{3(y^2 - 3x)} = \frac{3y - x^2}{y^2 - 3x}$$

Tangent Line Con't

- The slope of the tangent line is found by plugging (2,4) into the derivative

$$m = \frac{3y - x^2}{y^2 - 3x} = \frac{3(4) - (2)^2}{(4)^2 - 3(2)} = \frac{4}{5}$$

- Since we're looking for the tangent line:

$$y = \frac{4}{5}x + b$$

$$4 = \frac{4}{5}(2) + b$$

$$\frac{12}{5} = b$$

$$y = \frac{4}{5}x + \frac{12}{5}$$

6.5 – Related Rates

- Time is often present implicitly in a model. This means that the derivative with respect to time must be found implicitly.

Related Rates

- Suppose x and y are both functions of t (time), and that x and y are related by $xy^2 + y = x^2 + 17$
- When $x = 2$, $y = 3$, and $dx/dt = 13$.
- Find the value of dy/dt at that moment

Continued

- $xy^2 + y = x^2 + 17$
- Product and chain rules

$$x\left(2y\frac{dy}{dt}\right)+y^2\frac{dx}{dt}+\frac{dy}{dt}=2x\frac{dx}{dt}$$

- Now substitute $x = 2$, $y = 3$ and $dx/dt = 13$

$$2\left(6\frac{dy}{dt}\right)+9(13)+\frac{dy}{dt}=4(13)$$

Continued

$$2\left(6\frac{dy}{dt}\right)+9(13)+\frac{dy}{dt}=4(13)$$

$$12\frac{dy}{dt}+117+\frac{dy}{dt}=52$$

- Solve for dy/dt

$$13\frac{dy}{dt}=-65$$

$$\frac{dy}{dt}=-5$$

Solving the Problems

SOLVING A RELATED RATE PROBLEM

1. Identify all given quantities, as well as the quantities to be found. Draw a sketch when possible.
2. Write an equation relating the variables of the problem.
3. Use implicit differentiation to find the derivative of both sides of the equation in Step 2 with respect to time.
4. Solve for the derivative giving the unknown rate of change and substitute the given values.

Area Example

- A small rock is dropped into a lake. Circular ripples spread out over the surface of the water, with the radius of each circle increasing at a rate of 3/2 ft. per second.
- Find the rate of change of the area inside the circle formed by a ripple at the instant r = 4ft.

Area Example

- Area and radius are related by $A = \pi r^2$
- Take the derivative of each side with respect to time

$$\frac{d}{dt}A = \frac{d}{dt}\left(\pi r^2\right)$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

- Since the radius is increasing at the rate of 3/2 ft per second

$$\frac{dr}{dt} = \frac{3}{2}$$

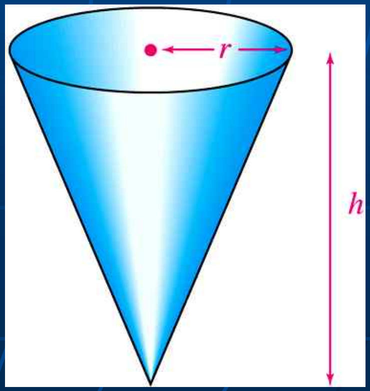
Area Continued

- Find the rate of change of the area inside the circle formed by a ripple at the instant r = 4ft.

$$\frac{dA}{dt} = 2\pi(4)\frac{3}{2} \approx 37.3 ft^2 \text{ per second}$$

One More--Volume

- A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2cm/hr, while the length is increasing at a rate of 0.8cm/hr. If the icicle is currently 4cm in radius, and 20cm long, is the volume of the icicle increasing or decreasing, and at what rate?



Volume

- The volume of a cone is found by

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{d}{dt}V = \frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right)$$

A constant is being multiplied to 2 variables—use the product rule & chain rule

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[r^2\frac{dh}{dt} + h(2r)\frac{dr}{dt}\right]$$

Volume

- The radius of the icicle is decreasing at a rate of 0.2cm/hr, while the length is increasing at a rate of 0.8cm/hr.

$$\frac{dr}{dt} = -0.2 \qquad \frac{dh}{dt} = 0.8$$

Volume

- The icicle is currently 4cm in radius and 20cm long. Plugging in we have:

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[r^2\frac{dh}{dt} + h(2r)\frac{dr}{dt}\right]$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[4^2(0.8) + (20)(8)(-0.2)\right]$$

$$\frac{dV}{dt} \approx -20$$

- The volume is decreasing at a rate of about $20\text{cm}^3/\text{hr}$

6.6
Differentials: Linear Approximation

DIFFERENTIALS

For a function $y = f(x)$ whose derivative exists, the **differential** of x, written dx, is an arbitrary real number (usually small compared with x); the **differential** of y, written dy, is the product of $f'(x)$ and dx, or

$$dy = f'(x)dx.$$

6.6
Differentials: Linear Approximation

- Find dy for $y = 6x^2$

- $dy/dx = 12x$
- $dy = 12x\,dx$

One More

- Find dy:

$$y = 800x^{\frac{-3}{4}} \text{ when } x = 16,\ dx = 0.01$$

$$dy = -600x^{\frac{-7}{4}}dx$$

$$= -600(16)^{\frac{-7}{4}}(0.01) \approx -0.046875$$

Linear Approximation

- Approximate $\sqrt{50}$

12.7
L'Hospital's Rule

- We started this semester with a discussion of limits:

$$\lim_{x \to 1} \frac{x^2+1}{x+4} = \frac{2}{5}$$

- On another note:

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x \to 1} (x+1) = 2$$

12.7
L'Hospital's Rule

- Looking at $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \dfrac{0}{0}$

- This leads to a meaningless result and is called the indeterminate form

L'Hospital's Rule

- L'Hospital's rule gives us a quick way to decide whether a quotient with the indeterminate form has a limit.

USING L'HOSPITAL'S RULE

1. Be sure that $\lim_{x \to a} f(x)/g(x)$ leads to the indeterminate form $0/0$ or ∞/∞.
2. Take the derivatives of f and g separately.
3. Find the limit of $f'(x)/g'(x)$; this limit, if it exists, equals the limit of $f(x)/g(x)$.
4. If necessary, apply l'Hospital's rule more than once.

Do NOT use the quotient rule!

Example

$$\text{Find} \lim_{x\to 2} \frac{3x-6}{\sqrt{2+x}-2} = 12$$

- A quick check gives you 0/0

$$f(x) = 3x-6 \qquad g(x) = \sqrt{2+x}-2 = (2+x)^{\frac{1}{2}}-2$$

$$f'(x) = 3 \qquad g'(x) = \frac{1}{2}(2+x)^{\frac{-1}{2}}(1) = \frac{1}{2\sqrt{2+x}}$$

- By L'H: $\displaystyle\lim_{x\to 2} \frac{3}{\frac{1}{2\sqrt{2+x}}} = \frac{3}{\frac{1}{4}} = 12$

One More

- Find $\lim_{x \to 0} \frac{e^x - x - 1}{x^2}$

- Substitution gives 0/0, so apply L'H

$$\lim_{x \to 0} \frac{e^x - 1}{2x} = \frac{0}{0}$$

- Do it again!

$$\lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

Lab 3

- Do these questions to hand in next time:
- 1. #54, P375 10. #15, P396
- 2. #1, P382 11. #9, P395
- 3. #4 & 7, P372 12. #9, P383
- 4. #3, P401 13. #23, P384
- 5. #28, P401 14. #19, P401
- 6. #20, P372 15. #29, P411
- 7. #32, P385 16. #43, P374
- 8. #3, P409 17. #26, P411
- 9. #11, P409 18-20. #36, 40, 45 P772