Chapter 5
Graphs and the Derivative

JMerrill, 2009

Chapter 4 Review

- Find f'(x) if $f(x) = (3x - 2x^2)^3$

- $3(3x - 2x^2)^2(3 - 4x)$

Review

$$\text{Find } f'(x) \text{ if } f(x) = \sqrt[3]{(x^2-1)^2}$$

$$\text{Rewrite: } \left(x^2-1\right)^{\frac{2}{3}}$$

$$\frac{2}{3}\left(x^2-1\right)^{\frac{-1}{3}}(2x)$$

$$\frac{4x}{3\sqrt[3]{\left(x^2-1\right)}}$$

Review

- Find f'(x) if $f(x)=x^2\sqrt{1-x^2}$

$$=x^2\left(1-x^2\right)^{\frac{1}{2}}$$

$$f'(x)=x^2\frac{d}{dx}\left[\left(1-x^2\right)^{\frac{1}{2}}\right]+\left(1-x^2\right)^{\frac{1}{2}}\frac{d}{dx}\left[x^2\right]$$

$$=x^2\left[\frac{1}{2}\left(1-x^2\right)^{\frac{-1}{2}}(-2x)\right]+\left(1-x^2\right)^{\frac{1}{2}}(2x)$$

$$=x^2\left(\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\right)+2x\sqrt{1-x^2}$$

Continued

$$= x^2\left(\frac{-x}{\sqrt{1-x^2}}\right) + 2x\sqrt{1-x^2}$$

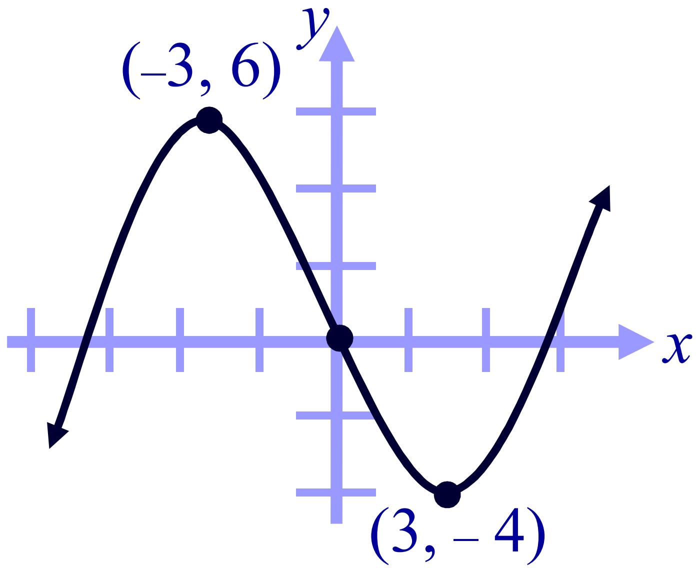
$$= \left(\frac{-x^3}{\sqrt{1-x^2}}\right) + 2x\sqrt{1-x^2}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)$$

$$= \left(\frac{-x^3}{\sqrt{1-x^2}}\right) + \left(\frac{2x\left(1-x^2\right)}{\sqrt{1-x^2}}\right)$$

$$= \frac{-x^3 + 2x - 2x^3}{\sqrt{1-x^2}} = \frac{-3x^3 + 2x}{\sqrt{1-x^2}} = \frac{x\left(-3x^2 + 2\right)}{\sqrt{1-x^2}}$$

5.1 Increasing & Decreasing Functions

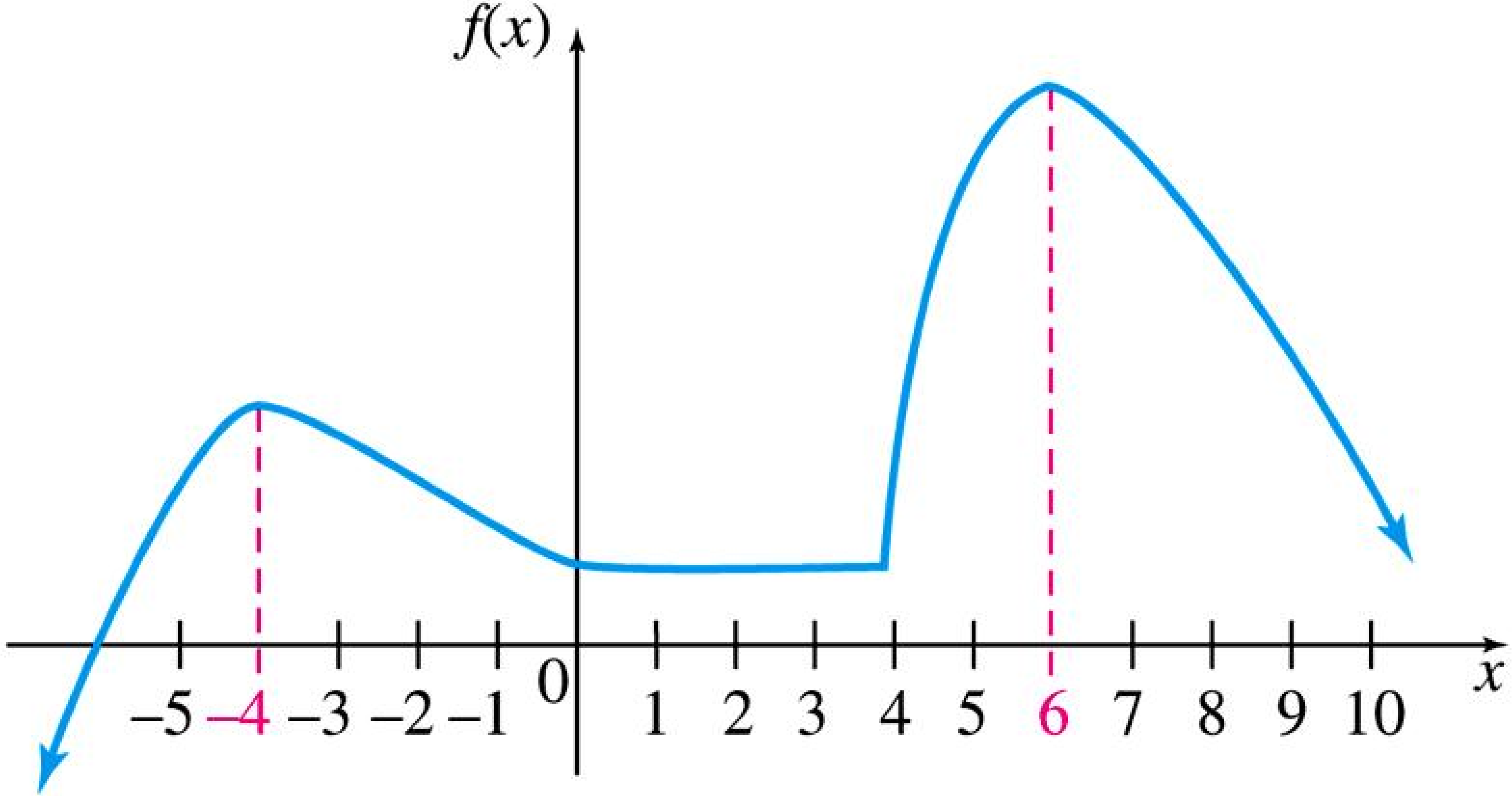
The graph of $y = f(x)$:



- **increases** on $(-\infty, -3)$,
- **decreases** on $(-3, 3)$,
- **increases** on $(3, \infty)$.

- A function can be increasing, decreasing, or constant

Increasing/Decreasing



Increasing/Decreasing Test

TEST FOR INTERVALS WHERE $f(x)$ IS INCREASING AND DECREASING

Suppose a function f has a derivative at each point in an open interval; then
if $f'(x) > 0$ for each x in the interval, f is *increasing* on the interval; $\nearrow$
if $f'(x) < 0$ for each x in the interval, f is *decreasing* on the interval; $\searrow$
if $f'(x) = 0$ for each x in the interval, f is *constant* on the interval. $\rightarrow$

Critical Numbers

CRITICAL NUMBERS

The **critical numbers** for a function f are those numbers c in the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist. A **critical point** is a point whose x-coordinate is the critical number c, and whose y-coordinate is $f(c)$.

The derivative can change signs (positive to negative or vice versa) where f'(c) = 0 or where f'(c) DNE. A critical point is a point whose x-coordinate is the critical number.

Applying the Test

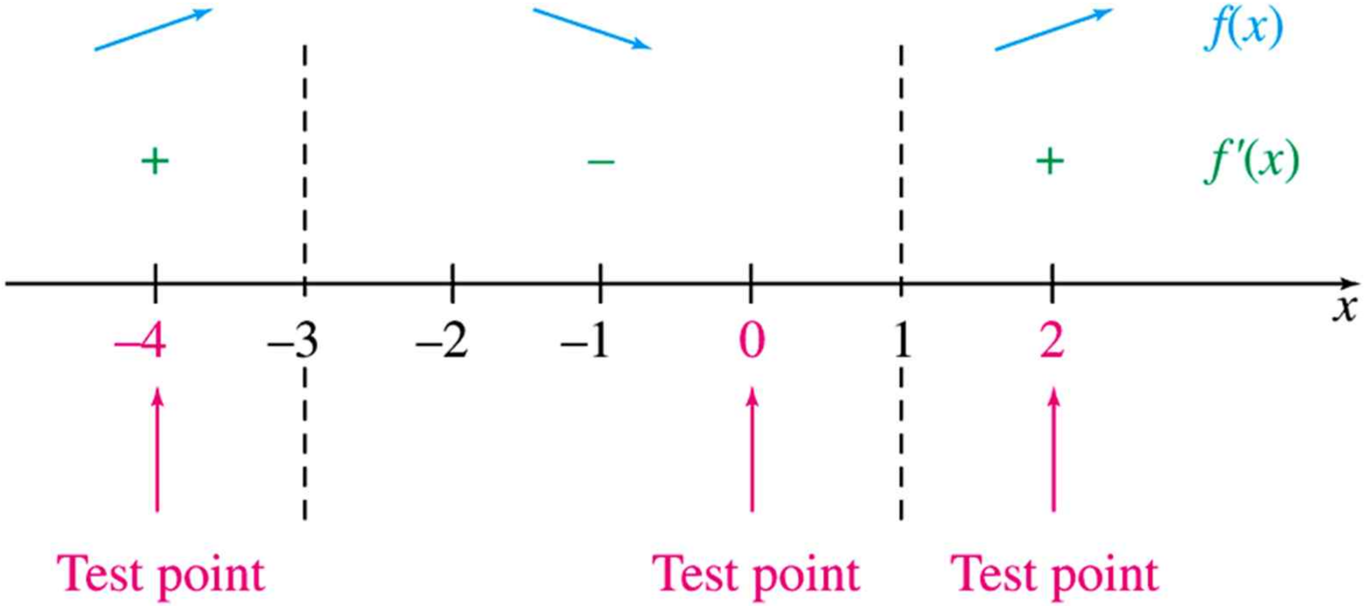
- Find the intervals where the function $f(x)=x^3+3x^2-9x+4$ is increasing/decreasing and graph.
- 1. Find the critical numbers by setting $f'(x) = 0$ and solving: $f'(x) = 3x^2 + 6x – 9$

$$0 = 3(x^2 + 2x – 3)$$
$$0 = 3(x + 3)(x - 1)$$
$$x = -3, 1$$

The tangent line is horizontal at x = -3, 1

Test

- Mark the critical points on a number line and choose test points in each interval

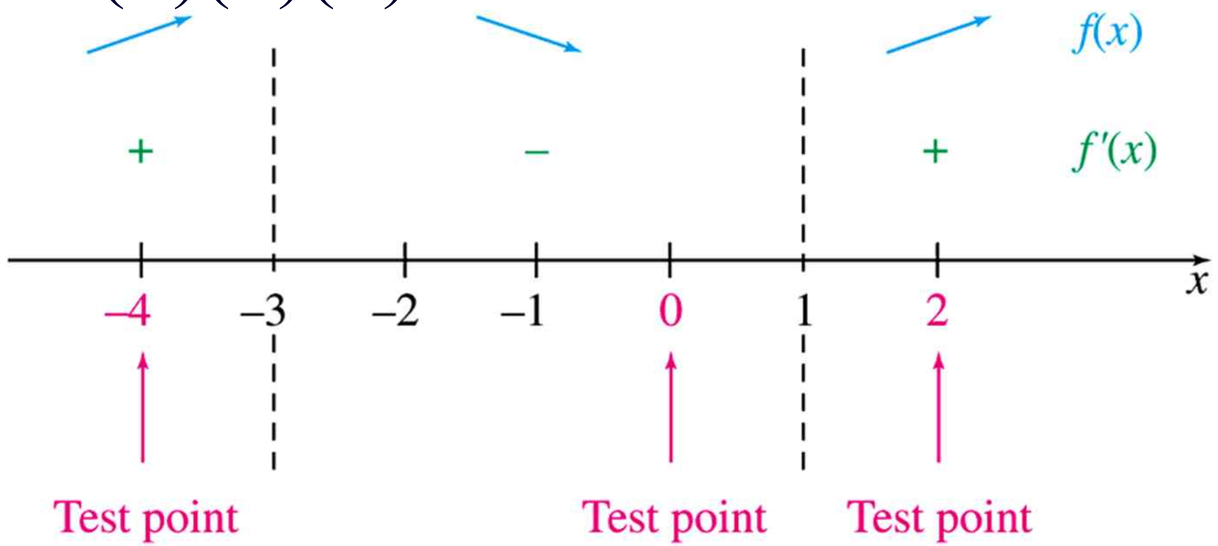


Test

- Evaluate the test points in f'(x): 3(x+3)(x-1)
- f'(-4) = (+)(-)(-) = +
- f'(0) = (+)(+)(-) = -
- f'(2) = (+)(+)(+) = +

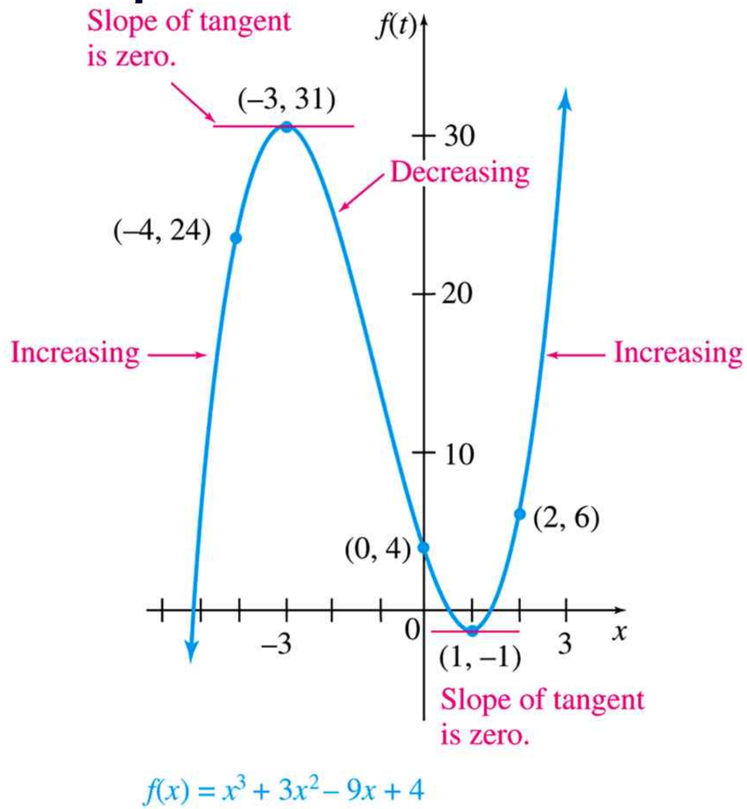
$\uparrow (-\infty,-3),(1,\infty)$

$\downarrow (-3,1)$

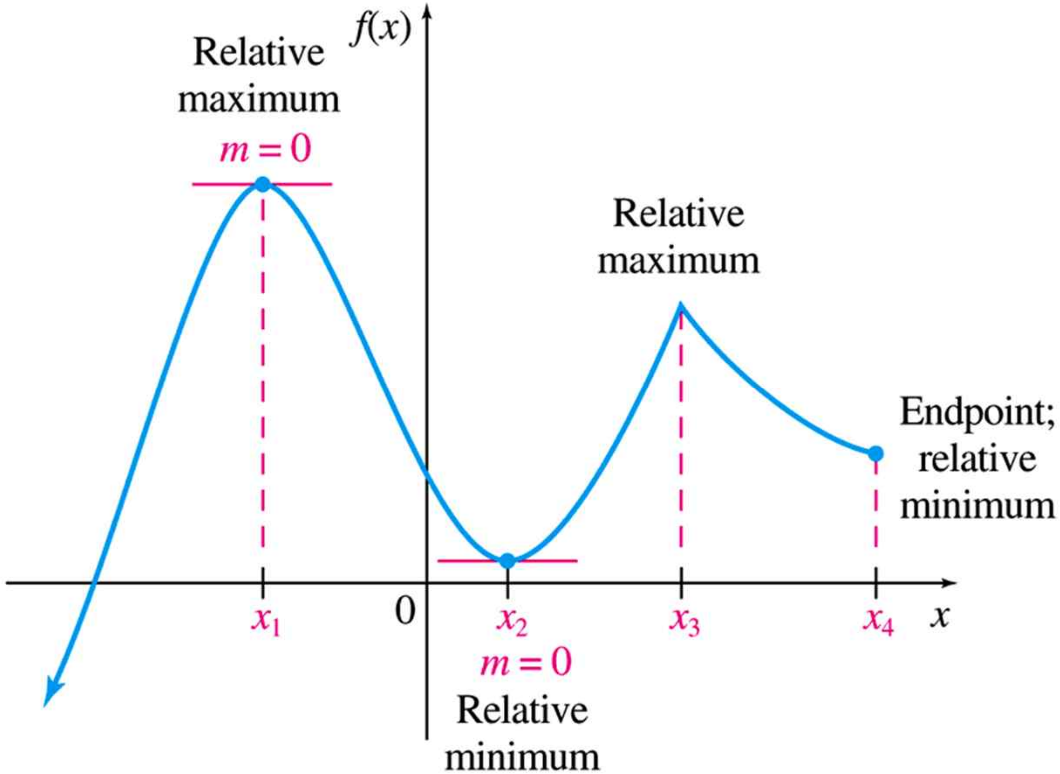


To Graph

- To graph the function, plug the critical points into f(x) to find the ordered pairs:
- $f(x)=x^3+3x^2-9x+4$
- $f(-3) = 31$
- $f(1) = -1$

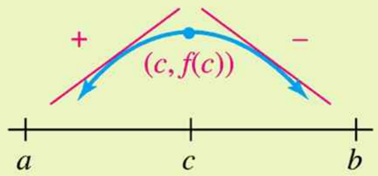
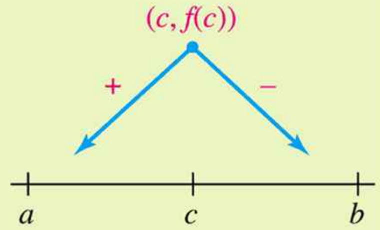
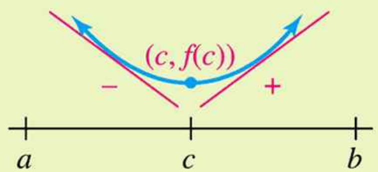
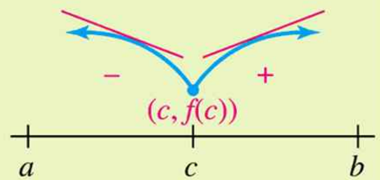
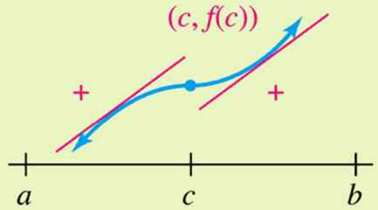
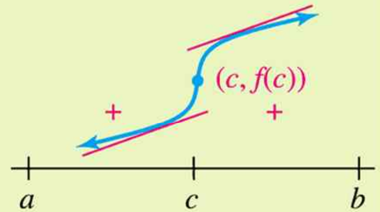
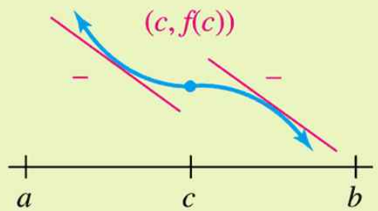
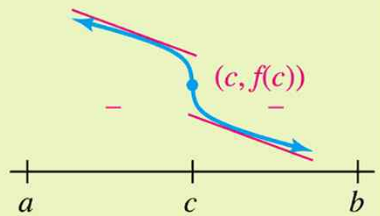


5.2 Relative (or Local) Extrema



If a function f has a relative extremum at c, then c is a critical number or c is an endpoint of the domain.

The First Derivative Test

$f(x)$ has:	Sign of f' in (a, c)	Sign of f' in (c, b)	Sketches	
Relative maximum	$+$	$-$		
Relative minimum	$-$	$+$		
No relative extrema	$+$	$+$		
No relative extrema	$-$	$-$		

First Derivative Test

FIRST DERIVATIVE TEST

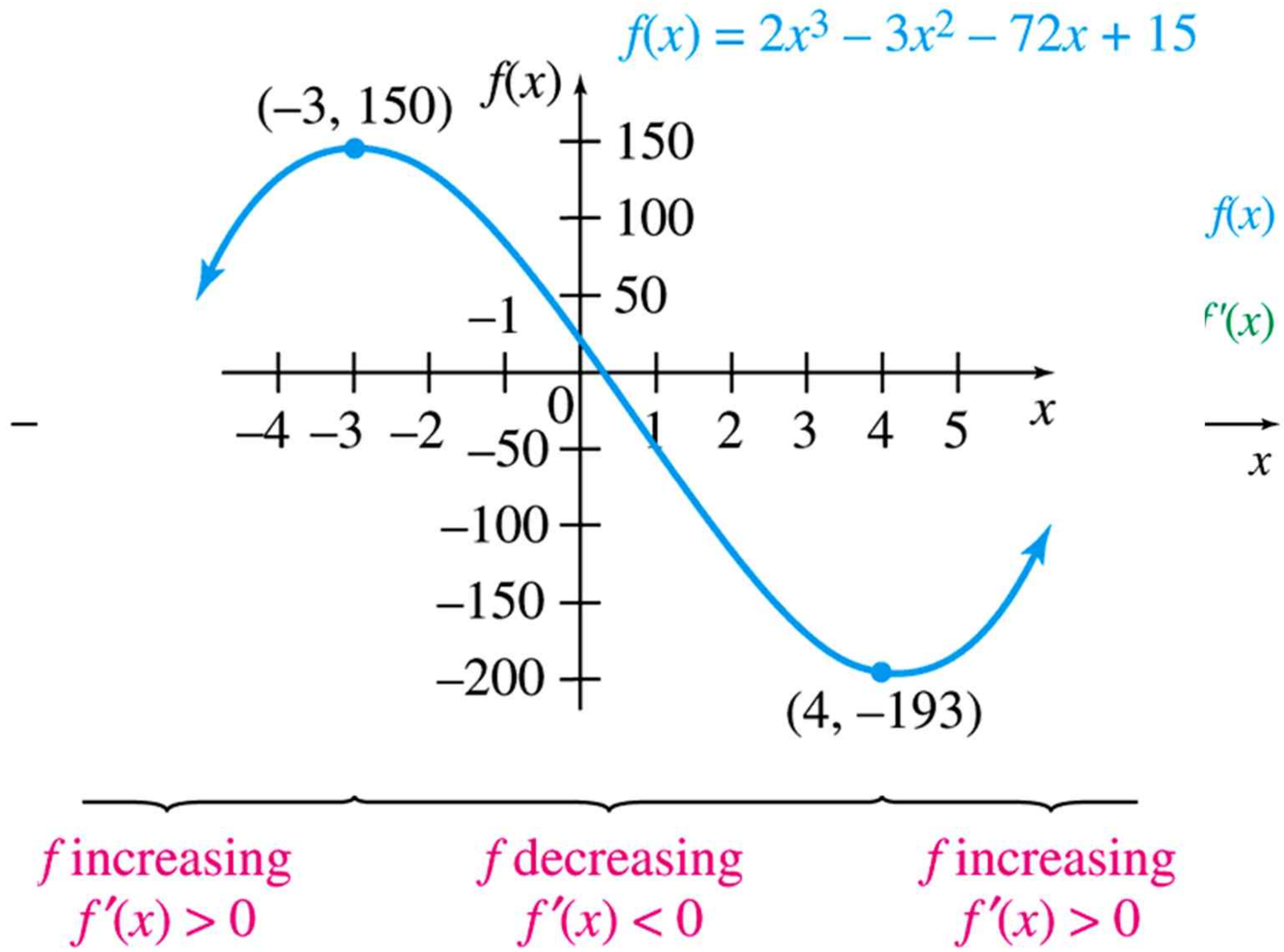
Let c be a critical number for a function f. Suppose that f is continuous on (a, b) and differentiable on (a, b) except possibly at c, and that c is the only critical number for f in (a, b).

1. $f(c)$ is a relative maximum of f if the derivative $f'(x)$ is positive in the interval (a, c) and negative in the interval (c, b).
2. $f(c)$ is a relative minimum of f if the derivative $f'(x)$ is negative in the interval (a, c) and positive in the interval (c, b).

You Do

- Find the relative extrema as well as where the function is increasing and decreasing and graph.
- $f(x) = 2x^3 - 3x^2 - 72x + 15$
- Critical points: $x = 4, -3$

You Do



$f(x) = 2x^3 - 3x^2 - 72x + 15$
(–3, 150)
f(x)
150
100
50
–1
0
–4 –3 –2
1 2 3 4 5
x
–50
–100
–150
–200
(4, –193)
f(x)
f′(x)
–
x
f increasing
f′(x) > 0
f decreasing
f′(x) < 0
f increasing
f′(x) > 0

5.3 Higher Derivatives, Concavity, the Second Derivative Test

- Given $f(x) = x^4 + 2x^3 - 5x + 7$, find
- $f'(x) =$ $4x^3 + 6x^2 - 5$
- $f''(x) =$ $12x^2 + 12x$
- $f'''(x) =$ $24x + 12$
- $f^{(4)}(x) =$ 24

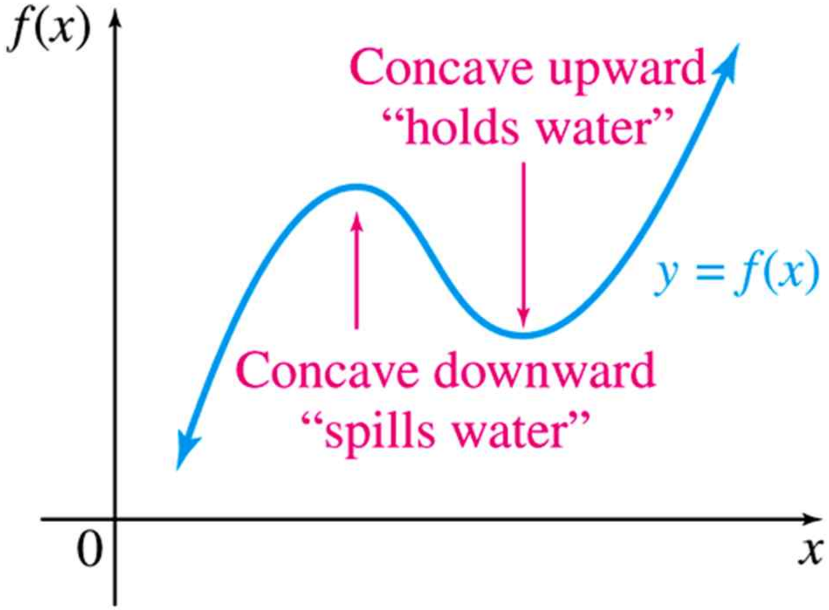
Find the 1st and 2nd Derivatives

- $f(x) = 4x(\ln x)$
- $f'(x) = 4x\left(\frac{1}{x}\right) + (\ln x)(4) = 4 + 4\ln x$
- $f''(x) = 0 + 4\frac{1}{x} = \frac{4}{x}$

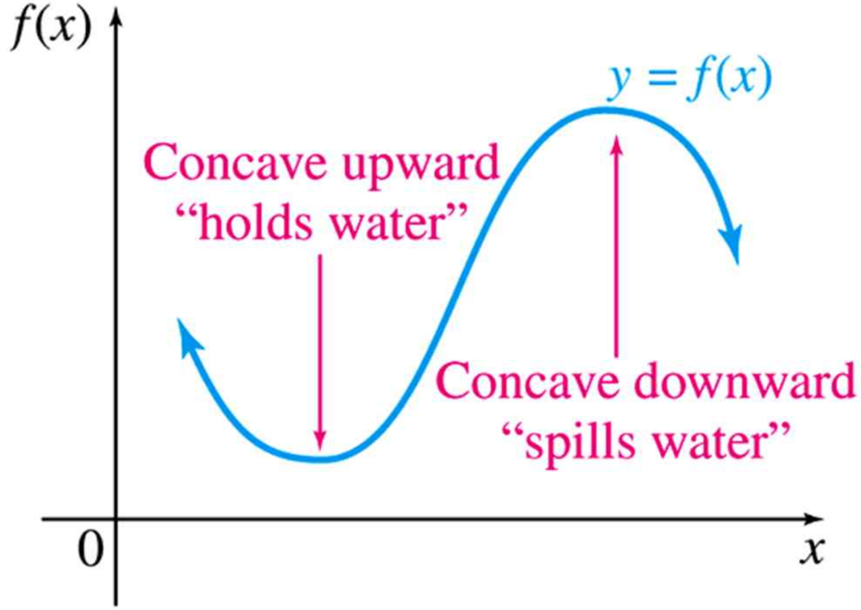
Position Function

- A car, moving in a straight line, starting at time, t, is given by $s(t) = t^3 - 2t^2 - 7t + 9$. Find the velocity and acceleration
- $v(t) = s'(t) = 3t^2 - 4t - 7$
- $a(t) = v'(t) = s''(t) = 6t - 4$

Concavity

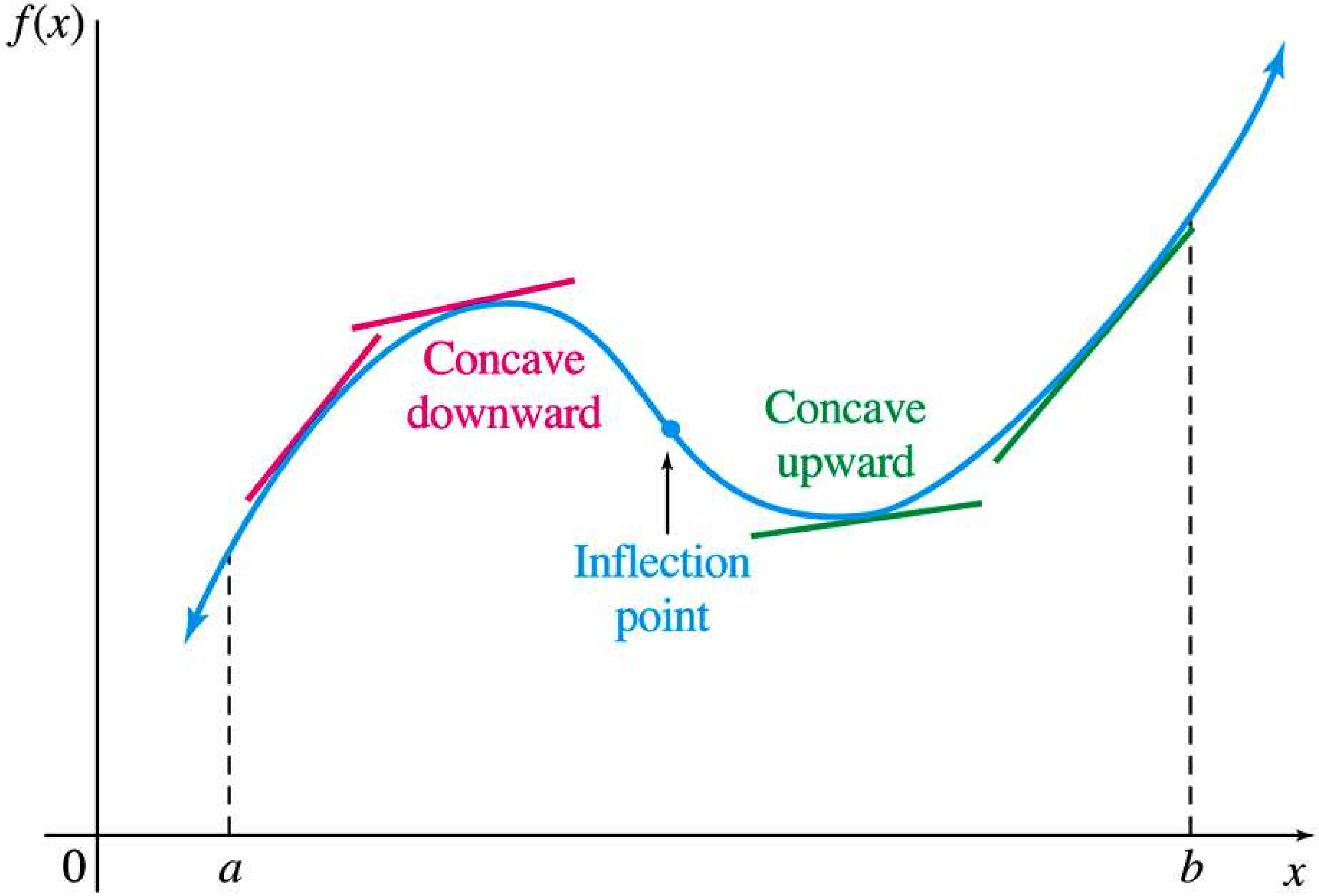


(a)



(b)

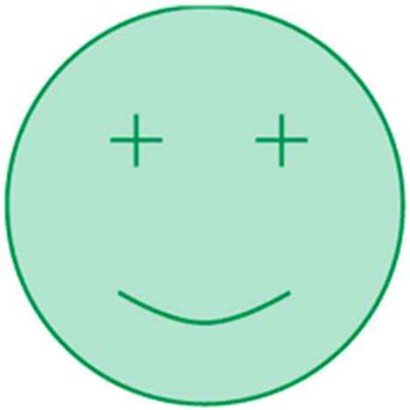
Concavity



Test for Concavity

TEST FOR CONCAVITY

Let f be a function with derivatives f' and f'' existing at all points in an interval (a, b). Then f is concave upward on (a, b) if $f''(x) > 0$ for all x in (a, b), and concave downward on (a, b) if $f''(x) < 0$ for all x in (a, b).



2^{nd} Derivative Test

SECOND DERIVATIVE TEST

Let f'' exist on some open interval containing c, and let $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$ or $f''(c)$ does not exist, then the test gives no information about extrema, so use the first derivative test.

By setting f''(x) = 0, you can fin the possible points of inflection (where the concavity changes).

At an inflection point for a function f, the second derivative is 0 or does not exist.

5.4 Curve Sketching

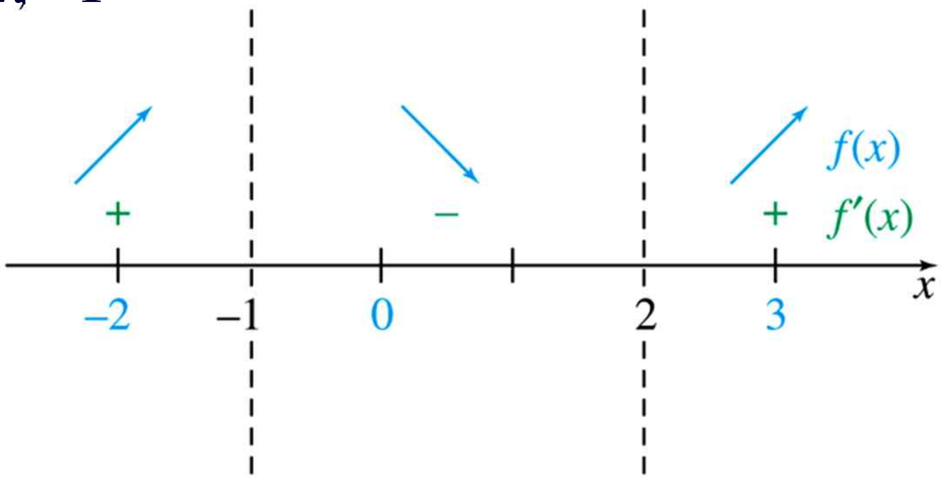
1. Note any restrictions in the domain (dividing by 0, square root of a negative number…)
2. Find the y-intercept (and x-intercept if it can be done easily).
3. Note any asymptotes (vertical asymptotes occur when the denominator = 0, horizontal asymptotes can be found by evaluating x as $x \to \infty$ or as $x \to -\infty$

Curve Sketching

4. Find f'(x). Locate critical points by solving f'(x) = 0.
5. Find f''(x). Locate points of inflection by solving f''(x) = 0. Determine concavity.
6. Plot points.
7. Connect the points with a smooth curve. Points are not connected if the function is not defined at a certain point.

Example

- Graph $f(x) = 2x^3 - 3x^2 - 12x + 1$
- Y-intercept is (0,1)
- Critical points: $f'(x) = 6x^2 - 6x - 12$
  - $x = 2, -1$



Example

- Graph $f(x) = 2x^3 - 3x^2 - 12x + 1$
- Y-intercept is (0,1)
- Critical points: $f'(x) = 6x^2 - 6x - 12$
  - $x = 2, -1$
- Points of inflection: $f''(x) = 12x - 6$
  - $x = \frac{1}{2}$

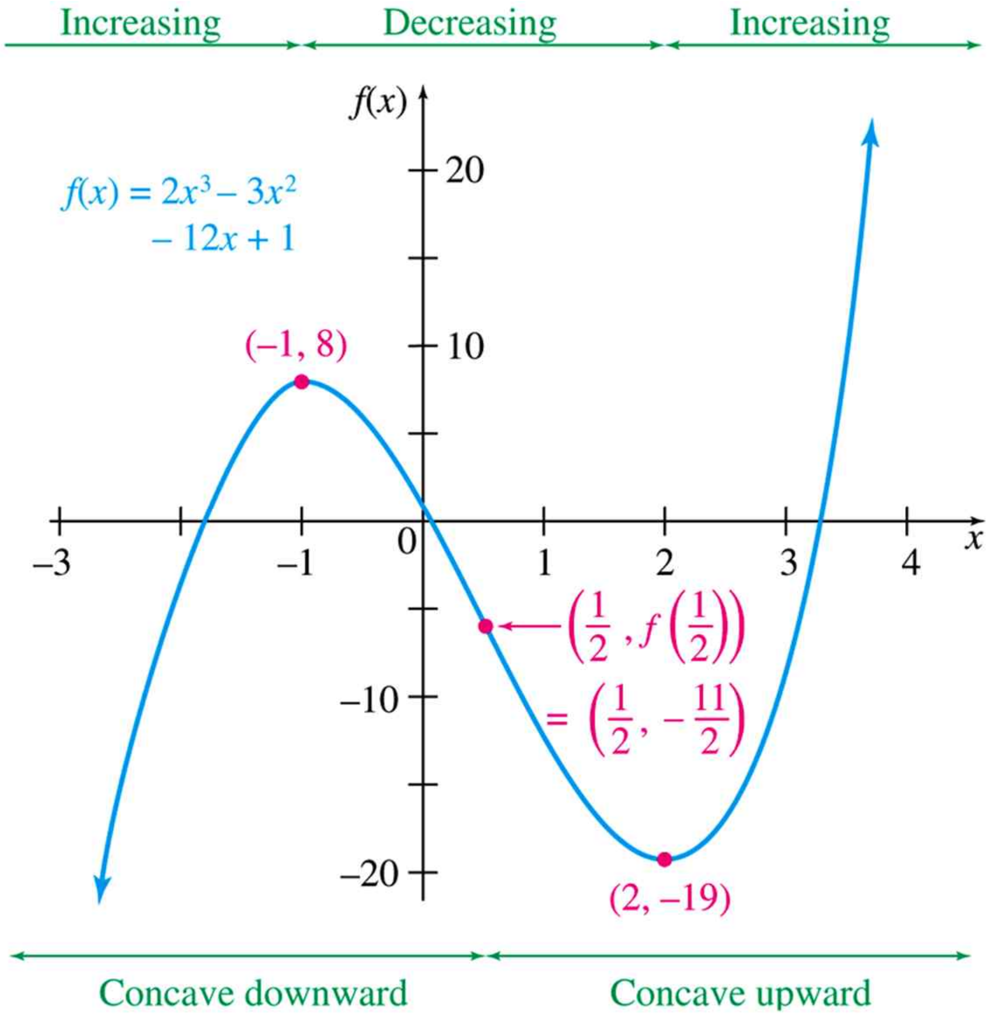
Example

Interval	$(-\infty, -1)$	$(-1, 1/2)$	$(1/2, 2)$	$(2, \infty)$
Sign of f'	$+$	$-$	$-$	$+$
Sign of f''	$-$	$-$	$+$	$+$
f Increasing or Decreasing	Increasing	Decreasing	Decreasing	Increasing
Concavity of f	Downward	Downward	Upward	Upward
Shape of Graph				

Example

- Graph $f(x) = 2x^3 – 3x^2 – 12x + 1$
- Y-intercept is (0,1)
- Critical points: $f'(x) = 6x^2 – 6x – 12$
  - $x = 2, -1$
- Points of inflection: $f''(x) = 12x – 6$
  - $x = -\frac{1}{2}$
- Plug critical points into f(x) to find y's
  - $f(2) = -19$, $f(-1) = 8$

Graph


Increasing
Decreasing
Increasing
f(x)
20
10
–10
–20
$f(x) = 2x^3 - 3x^2 - 12x + 1$
(–1, 8)
0
–3
–1
1
2
3
4
x
$\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, -\frac{11}{2}\right)$
(2, –19)
Concave downward
Concave upward