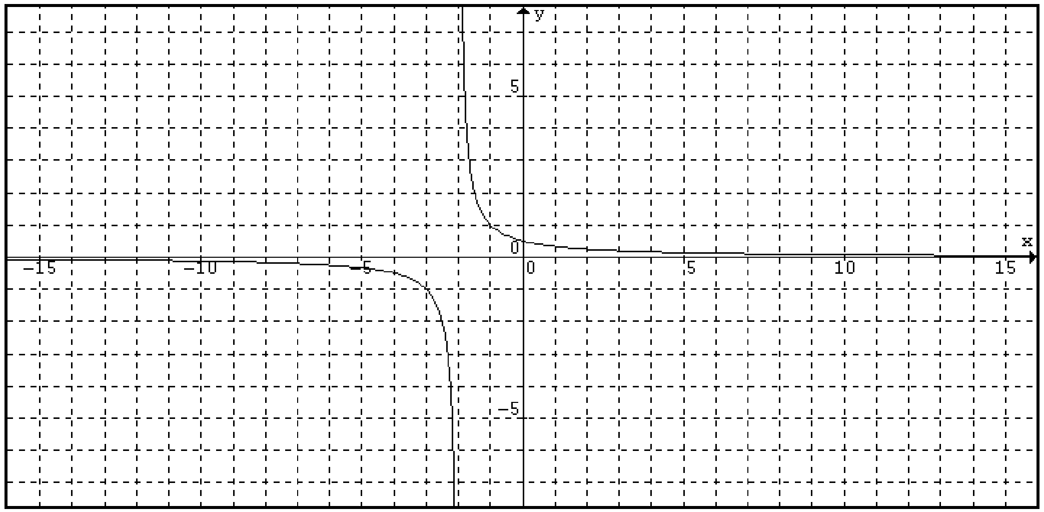
3.2 Continuity

JMerrill, 2009

Review 3.1

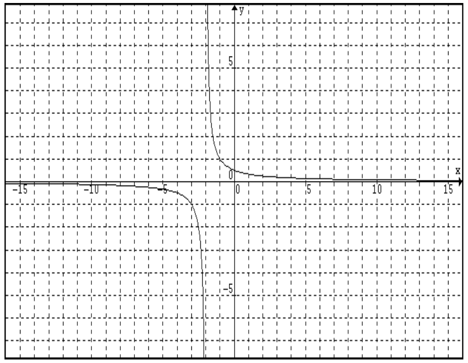
Find: $\lim_{x \to -2} f(x) = \dfrac{1}{x+2}$



Direct substitution causes division by zero. Factoring is not possible, so what are you going to do?

Review 3.1

x	f(x)
-3	-1
-2.5	-2
-2.1	-10
-2.01	-100
-2.001	-1000



x	f(x)
-1	1
-1.5	2
-1.9	10
-1.99	100
-1.999	1000

As x approaches -2 from the left,

f(x) approaches $-\infty$.

As x approaches -2 from the right,

f(x) approaches ∞.

Therefore, the limit DNE

Review – You Do

Describe what is happening to x and determine if a limit exists

x	f(x)
-4	-1.333
-3.5	-2.545
-3.1	-12.16
-3.01	-120.2
-3.001	-1200

x	f(x)
-2	1
-2.5	2.2222
-2.9	11.837
-2.99	119.84
-2.999	1199.8

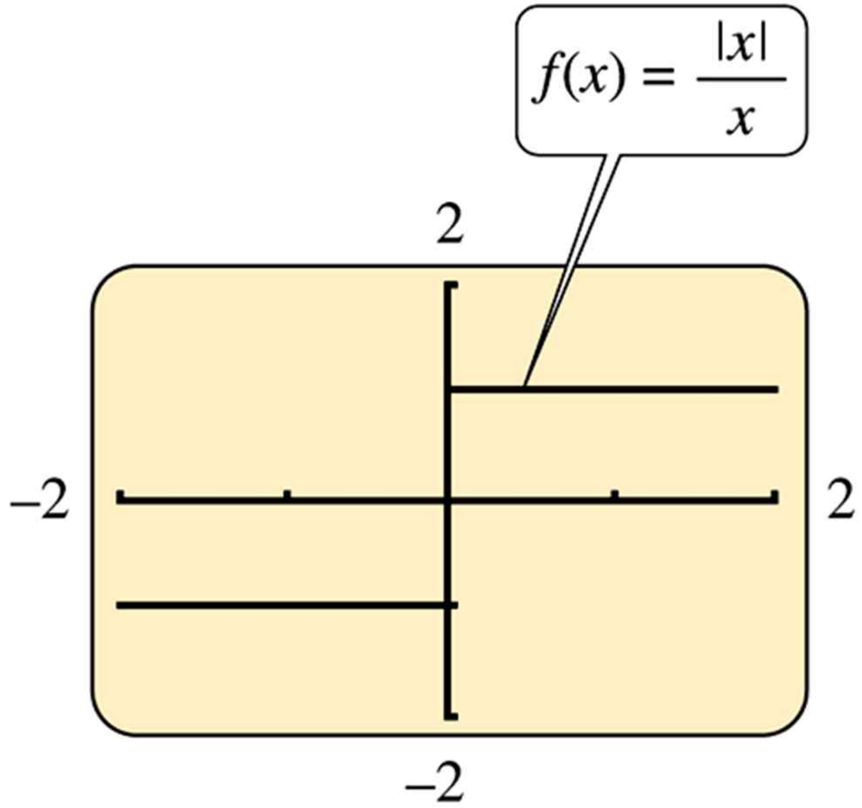
Therefore, the limit DNE.

As x approaches __-3__ from the __left__, f(x) approaches __$-\infty$__.

As x approaches __-3__ from the __right__, f(x) approaches __∞__.

Review

- Does the limit exist…



At 1?

At -1?

At 0?

3.2 Continuity

Continuous Functions

CONTINUITY AT $x = c$

A function f is **continuous** at $x = c$ if the following three conditions are satisfied:

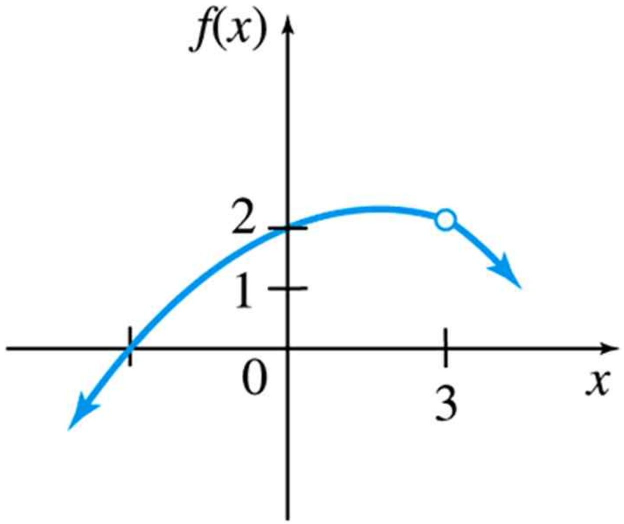
1. $f(c)$ is defined,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

If f is not continuous at c, it is **discontinuous** there.

- A function is continuous if you can draw the function without lifting your pencil.

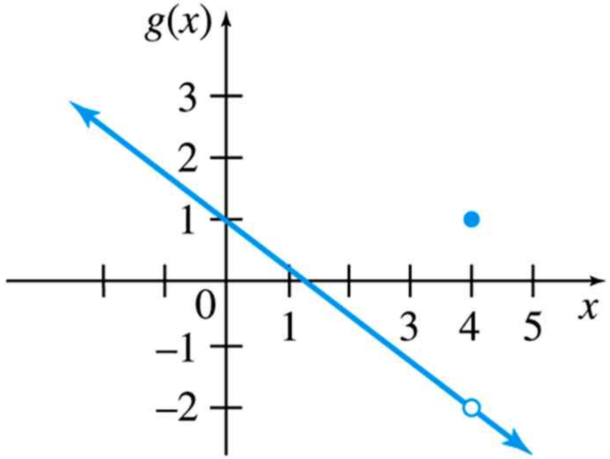
Continuous?

- Does the limit exist at x = 3? What is the limit?
- Is the function continuous?



Continuous?

- Is the function continuous?

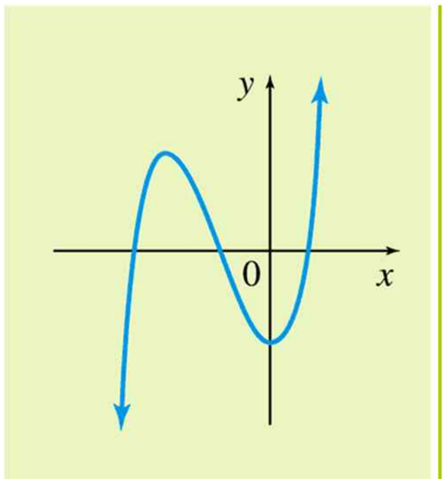


But, we can make the function continuous by defining g(4) = -2

- If a function can be made to be continuous by defining or redefining a single point, the function has a *removable discontinuity*.

Continuity on an Open Interval

- A function is continuous on an open interval if it is continuous at every x-value in the interval.



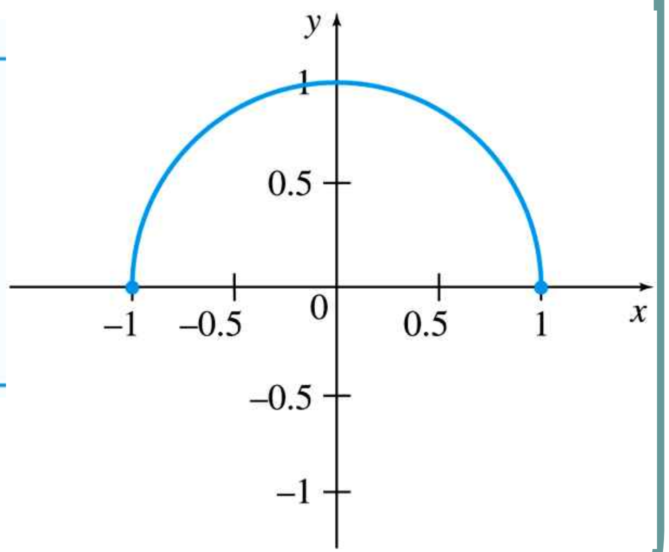
Continuity on a Closed Interval

- If you have a closed interval, then we have to define the endpoints.

CONTINUITY ON A CLOSED INTERVAL

A function is **continuous on a closed interval** $[a, b]$ if

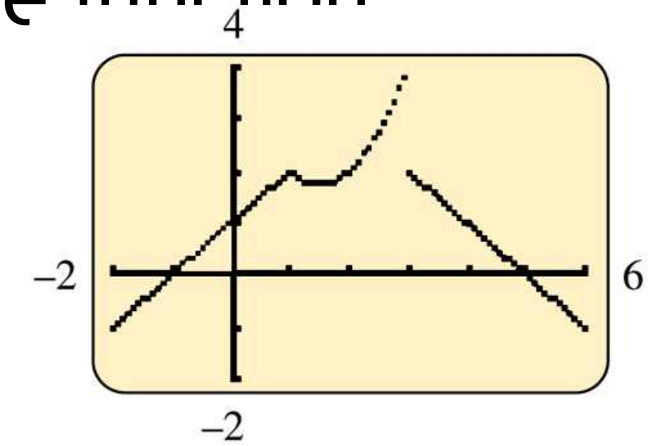
1. it is continuous on the open interval (a, b),
2. it is continuous from the right at $x = a$, and
3. it is continuous from the left at $x = b$.



Graphs

- If functions are given by graphs, it's pretty easy to tell if they're continuous. However, some functions are not so easy to tell.
- Consider the piecewise function

$$f(x) = \begin{cases} x+1 & \text{if } x<1 \\ x^2-3x+4 & \text{if } 1 \le x \le 3 \\ 5-x & \text{if } x>3 \end{cases}$$



3.3 Rates of Change

- One of the main applications of calculus is determining how a variable changes with respect to another variable.
- Average speed is a good example of average rate of change

$$\text{Average Speed} = \frac{\text{Distance}}{\text{Time}}$$

Average Rate of Change

AVERAGE RATE OF CHANGE

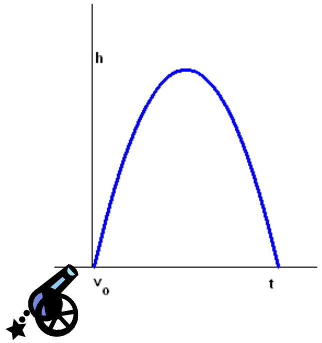
The **average rate of change** of $f(x)$ with respect to x for a function f as x changes from a to b is

$$\frac{f(b) - f(a)}{b - a}.$$

This book refers to this as the Difference Quotient

Average Rate Example

- Find the average rate of change for the function $f(x) = 3x^2 - 2$ from $x = 3$ to $x = 5$
- $f(b) = f(5) = 3(5)^2 - 2 = 73$
- $f(a) = f(3) = 3(3)^2 - 2 = 25$
- Evaluate, using the formula
- $$\frac{f(5)-f(3)}{5-3} = \frac{73-25}{2} = 24$$
- 24 is the average rate of change of the function from 3 to 5



You Do

- Find the average rate of change for the given function from x = 0 to 4
- $f(x) = -x^2 + 4$

$$\frac{f(b)-f(a)}{b-a}$$

$$f(4) = -(4)^2 + 4 = -12$$

$$f(0) = 0^2 + 4 = 4$$

$$\frac{-12-4}{4} = -4$$

Instantaneous Rate of Change

- The book uses:

INSTANTANEOUS RATE OF CHANGE

The **instantaneous rate of change** for a function f when $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists.

- We're going to use: $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Example: $x^2 - x + 4$

$$\lim_{h\to 0}\frac{\left[(x+h)^2-(x+h)+4\right]-f(x)}{h}$$

$$\lim_{h\to 0}\frac{\left[(x+h)^2-(x+h)+4\right]-(x^2-x+4)}{h}$$

$$\lim_{h\to 0}\frac{x^2+2hx+h^2-x-h+4-x^2+x-4}{h}$$

$$\lim_{h\to 0}\frac{2hx+h^2-h}{h} \qquad \lim_{h\to 0}\frac{h(2x+h-1)}{h}$$

$$\lim_{h\to 0}=2x+0-1=2x-1$$

You Do:

- Find the instantaneous rate of change of $f(x) = 3x^2 – x + 27$

$6x - 1$

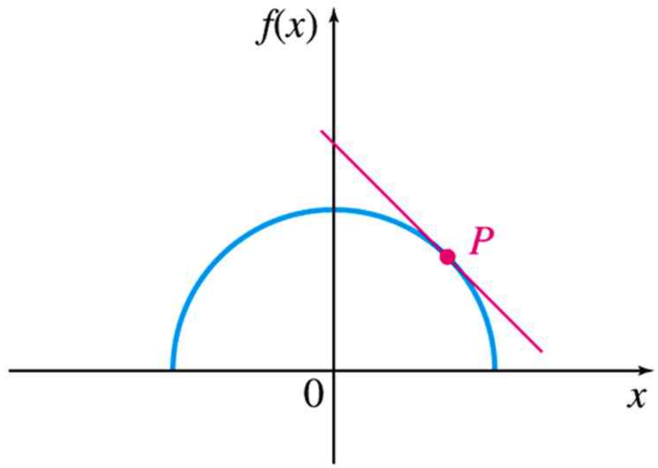
- Example from P.185

3.4

- Definition of the Derivative

Tangent Lines

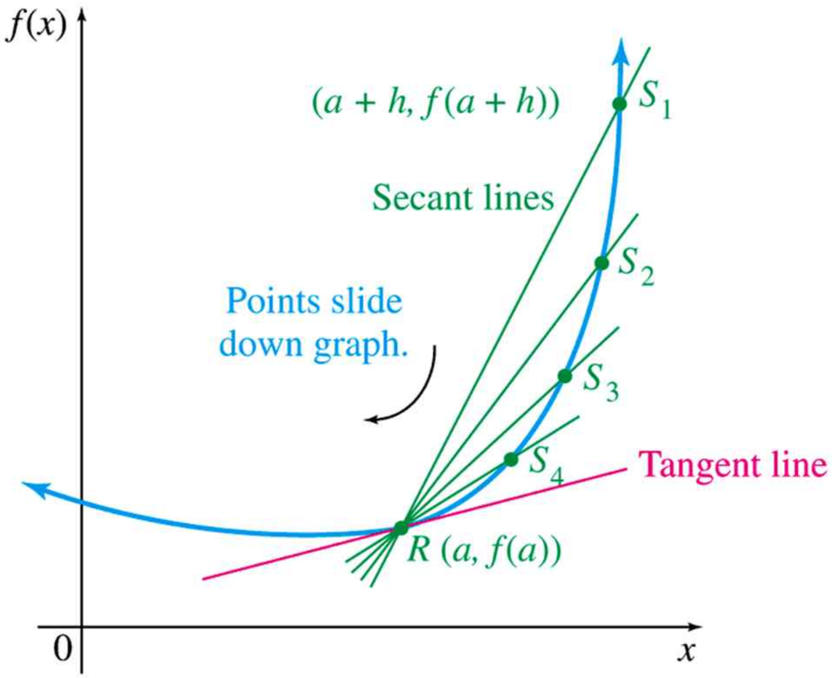
- A tangent line is a line that touches a curve at only one point.



- Since we can't find the slope of a curve, the slope of the tangent line is the slope of the curve at that point.

Secant Lines

- A secant line is a line going through the tangent line point on the curve and another point.
- As the fixed point S's get closer and closer to R, the slopes of the secant lines approach a limit as h approaches 0. The limit is the slope of the tangent line, which is the instantaneous rate of change.



Find the Tangent to the Curve

- Find the tangent to the curve $f(x) = x^2 + 2$ at $x = -1$
- A. Find the equation of the tangent line

SLOPE OF THE TANGENT LINE

The **tangent line** of the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through this point having slope

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists. If this limit does not exist, then there is no tangent at the point.

Example

- Find the tangent to the curve $f(x) = x^2 + 2$ at $x = -1$
- Find the ordered pair where the tangent line touches the curve:
- Since $x = -1$, $y = (-1)^2 + 2 = 3$
- The tangent line touches the curve at $(-1, 3)$

Example

- The limit is the first step to finding the slope of the tangent line:

$$\lim_{h\to 0}\frac{(x+h)^2+2-(x^2+2)}{h}$$

$$\lim_{h\to 0}\frac{x^2+2xh+h^2+2-x^2-2}{h}$$

$$\lim_{h\to 0}\frac{h(2x+h)}{h}=2x$$

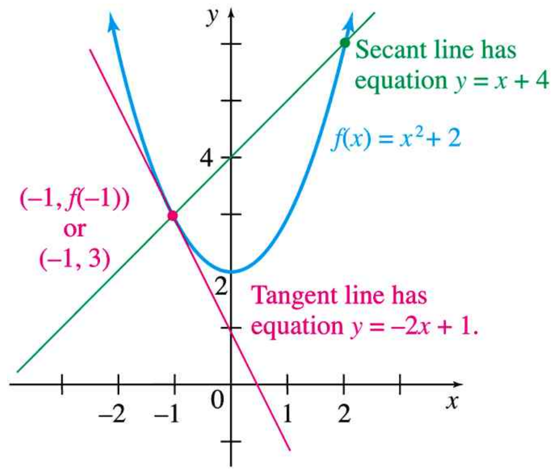
Plug the x-value into the limit answer: 2(-1) = -2.

m = -2

Finish

- We know the tangent line has m = -2 and touches the curve at (-1, 3)
- Write the equation:
- y = mx + b
- 3 = (-2)(-1) + b
- 3 = 2 + b
- 1 = b

The equation of the tangent line is y = -2x + 1



The Derivative

- The notation f'(x) is called the derivative with respect to x

DERIVATIVE

The **derivative** of the function f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided this limit exists.

The Derivative

The Difference Quotient and the Derivative

Difference Quotient	Derivative
$$\frac{f(b) - f(a)}{b - a}$$	$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
Slope of the secant line	Slope of the tangent line
Average rate of change	Instantaneous rate of change
Average velocity	Instantaneous velocity
Average rate of change in cost, revenue, or profit	Marginal cost, revenue, or profit

The Derivative P. 209

EXISTENCE OF THE DERIVATIVE

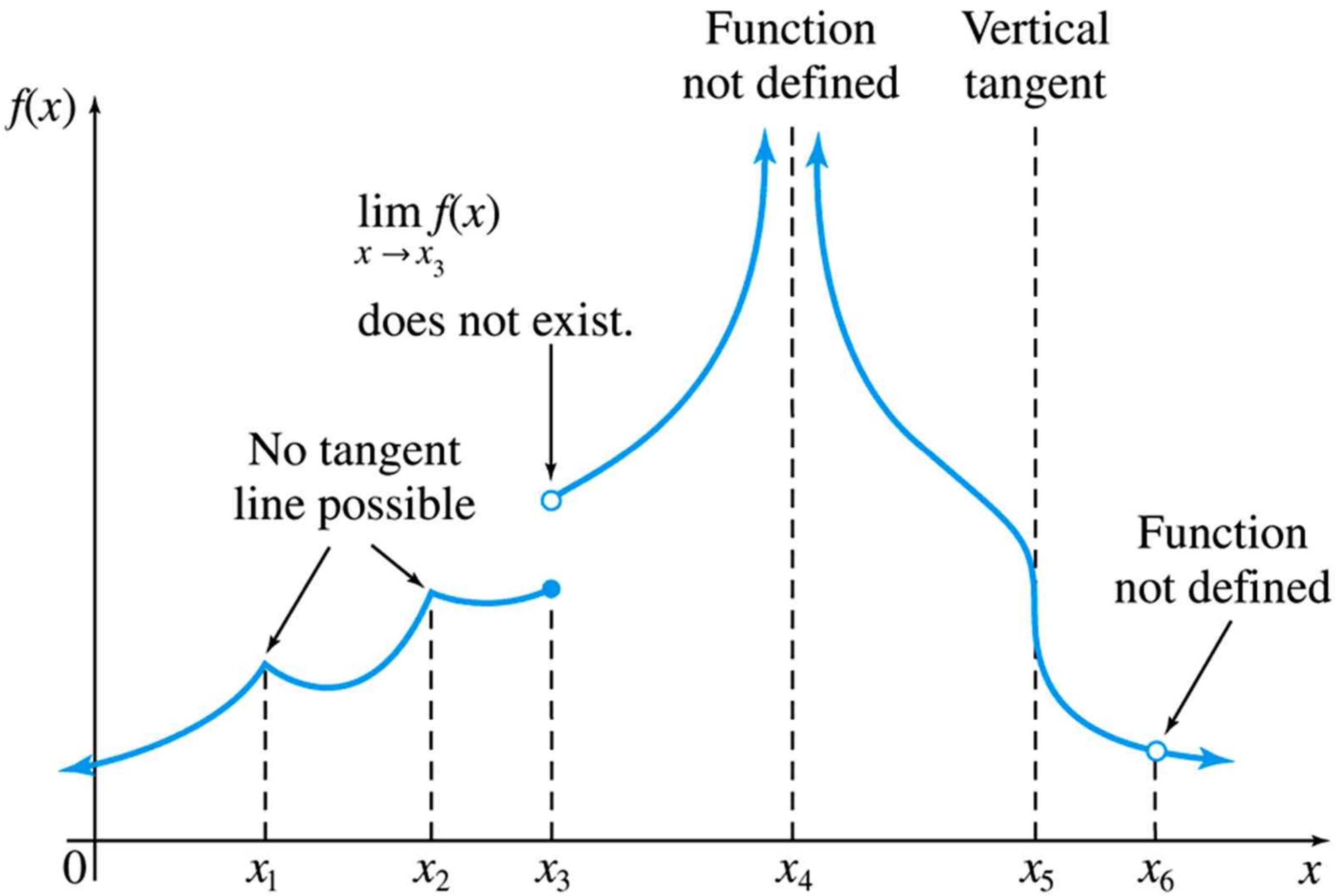
The derivative exists when a function f satisfies *all* of the following conditions at a point.

1. f is continuous,
2. f is smooth, and
3. f does not have a vertical tangent line.

The derivative does *not* exist when *any* of the following conditions are true for a function at a point.

1. f is discontinuous,
2. f has a sharp corner, or
3. f has a vertical tangent line.

The Derivative P. 209



To Find the Tangent to the Curve at a Point

- Substitute the x-value in the equation to find the corresponding y-value (if y is not given)
- Find $f'(x)$
- Substitute the x-value into $f'(x)$ to find the slope of the tangent line
- Now that you have a point and a slope, write the equation for the tangent line.
- $y = mx + b$

You Do

- Find the tangent to the curve $f(x) = 3x^2 - 6x + 2$ at $x = 2$
- $y = 6x - 10$