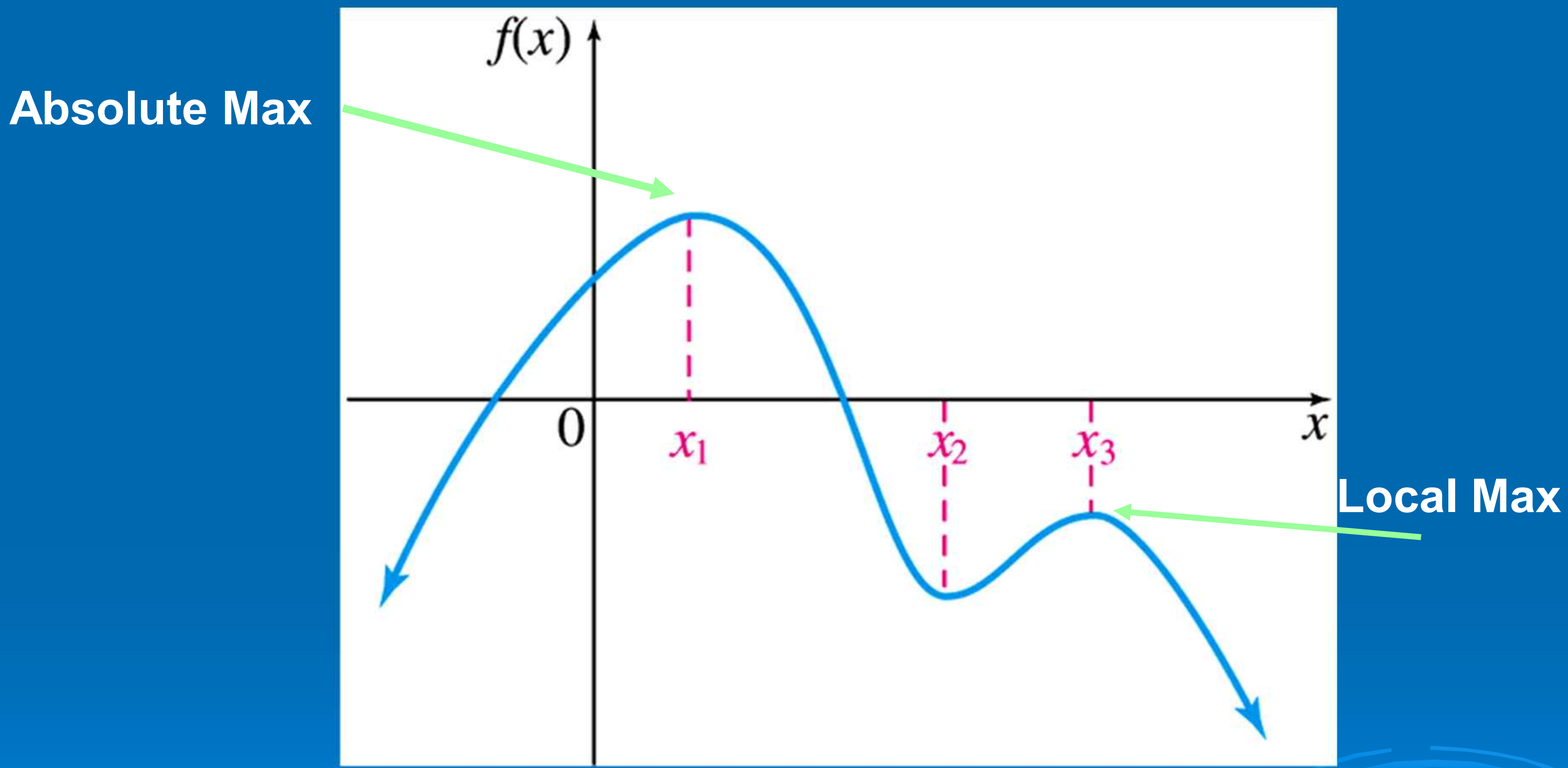
Chapter 6
Applications of the Derivative

JMerrill, 2009

6.1 – Absolute Extrema

- This chapter deals with the topic of optimization
- People always want to maximize income, minimize costs.
- These are examples of the use of a derivative—anytime we want to describe a rate of change.

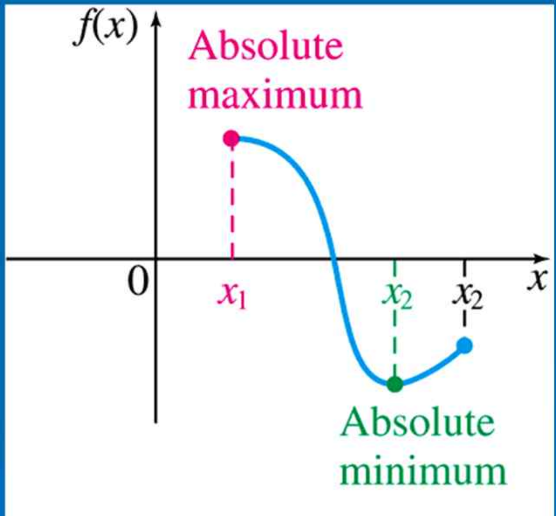
Absolute Maximum/Minimum



The max/min occurs at the x-value, but the actual max/min is the y-value

Max/Min

- Max/min can occur at endpoints



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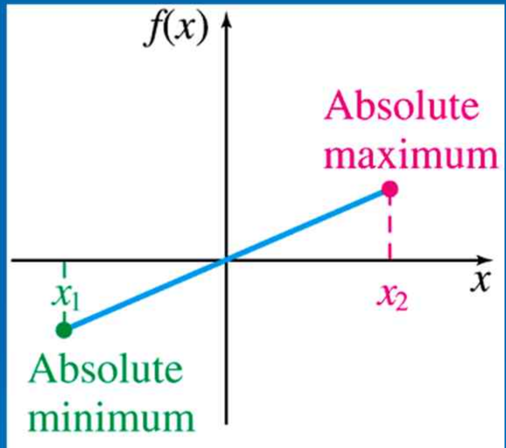

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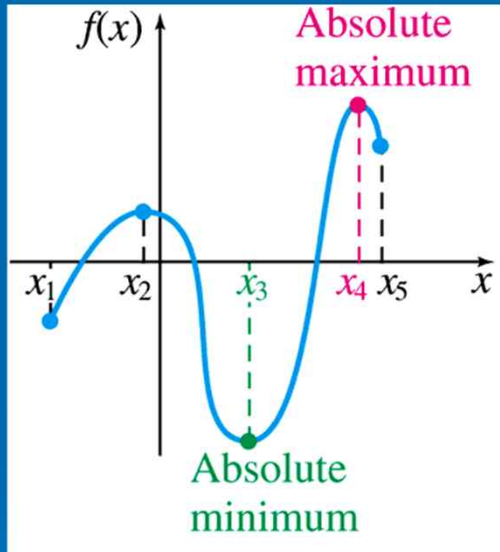

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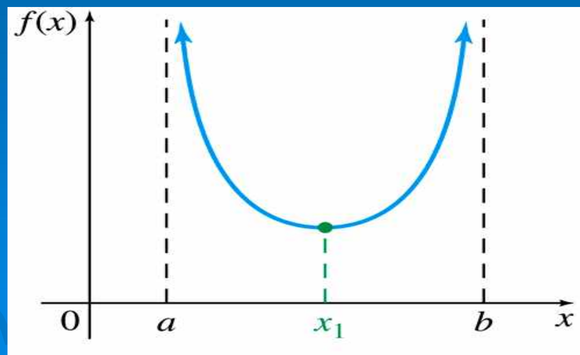
An absolute extrema is either the largest or smallest value in some interval (open or closed). A function can have only one absolute extrema.

Max/Min

EXTREME VALUE THEOREM

A function f that is continuous on a closed interval $[a, b]$ will have both an absolute maximum and an absolute minimum on the interval.

- This theorem guarantees the existence of absolute extrema on a closed interval. However, A continuous function on an open interval may or may not have an absolute max/min



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Finding Absolute Extrema

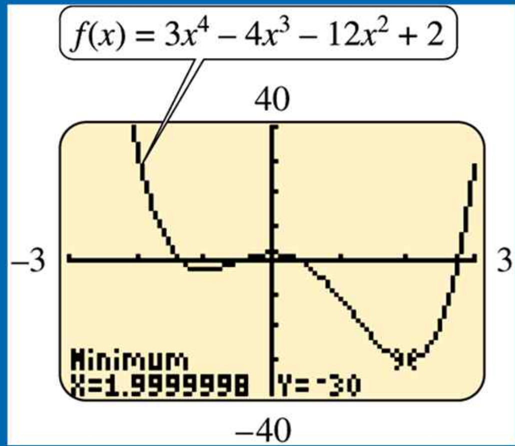
- To find absolute extrema:
  - Find the derivative
  - Find and evaluate the critical numbers in (a,b)
  - Evaluate the endpoints in [a,b]
  - The largest value is the absolute max; the smallest is the absolute min

Example

- Find the absolute extrema, if they exist, for the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 2$

- $f'(x) = 12x^3 - 12x^2 - 24x$
- $0 = 12(x^2 - x - 2)$
- $0 = 12x(x + 1)(x - 2)$
- $x = 0, -1, 2$
- There are no values where f'(x) does not exist, so evaluate the critical numbers.

Example Con't



- f(-1) = -3
- f(0) = 2
- f(2) = -30 **Absolute Min**
- This is an open interval so we cannot evaluate the endpoints. Instead we evaluate the limit of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 2$ as x becomes increasingly large ($x \to \infty$).
- Since the function grows without bound, the absolute min of -30 occurs at x = 2. Graphing proves this.

Critical Point Theorem

- In many applied extrema problems, a continuous function on an open interval has just one critical number

CRITICAL POINT THEOREM

Suppose a function f is continuous on an interval I and that f has exactly one critical number in the interval I, located at $x = c$.

If f has a relative maximum at $x = c$, then this relative maximum is the absolute maximum of f on the interval I.

If f has a relative minimum at $x = c$, then this relative minimum is the absolute minimum of f on the interval I.

Poverty Example

- Based on the data from the US Census Bureau, the number of people (in millions) in the US below poverty level between 1999 and 2006 can be approximated by the function
- $p(t) = -0.0982t^3 + 1.210t^2 – 3.322t + 34.596$ where t is the number of years since March 1999.
- In what year did the number of people living below poverty level reach its absolute maximum?
- What was the maximum number of people below poverty level during this period?

Poverty

- For what interval is the function defined?
- [0,7]—Between 1999 and 2006
- Find the critical numbers:
  - $p'(t) = -0.2946t^2 + 2.420t - 3.322$
- Can't factor so use quadratic formula to find the critical numbers of $t = 1.74, 6.47$
- Because it is a closed interval, we must also evaluate the endpoints 0, 7

Poverty

t-Value	Value of Function	
0	34.6	
1.74	32.0	Absolute minimum
6.47	37.2	← Absolute maximum
7	36.9	

- About 6.47 years after March 1999 (fall, 2005) about 37.2 million people were below poverty level
- Also, about 32.0 million people living below poverty level was reached about 1.74 years after March 1999 (end of the year 2000)

6.2 Applications of Extrema

SOLVING AN APPLIED EXTREMA PROBLEM

1. Read the problem carefully. Make sure you understand what is given and what is unknown.
2. If possible, sketch a diagram. Label the various parts.
3. Decide on the variable that must be maximized or minimized. Express that variable as a function of *one* other variable.
4. Find the domain of the function.
5. Find the critical points for the function from Step 3.
6. If the domain is a closed interval, evaluate the function at the endpoints and at each critical number to see which yields the absolute maximum or minimum. If the domain is an open interval, apply the critical point theorem when there is only one critical number. If there is more than one critical number, evaluate the function at the critical numbers and also find the limit as the endpoints of the interval are approached to determine if an absolute maximum or minimum exists at one of the critical points.

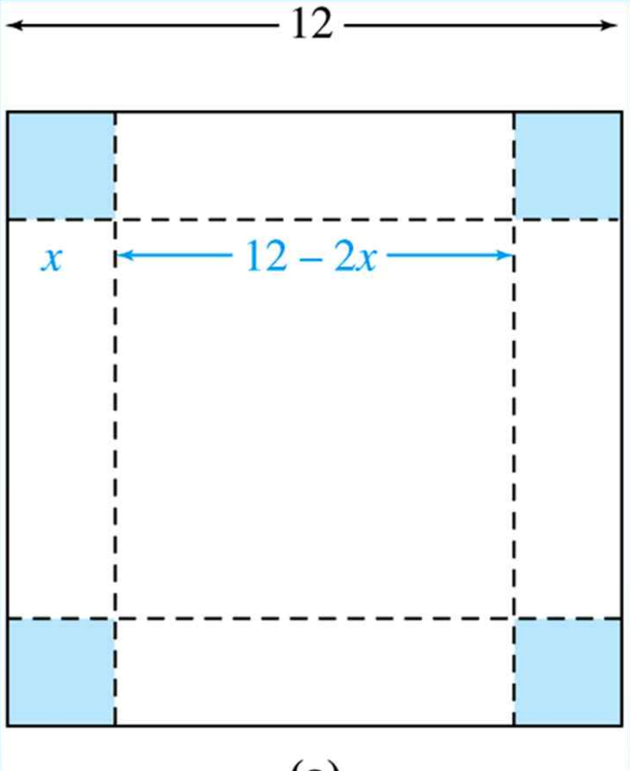
Maximizing Volume

- An open box is to be made from cutting a square from each corner of a 12-in. by 12-in. piece of metal and then folding up the sides. What size square should be cut from each corner to produce a box of maximum volume?

- 1. Known: 12 x 12 in. sides

Maximizing Volume

- 2.



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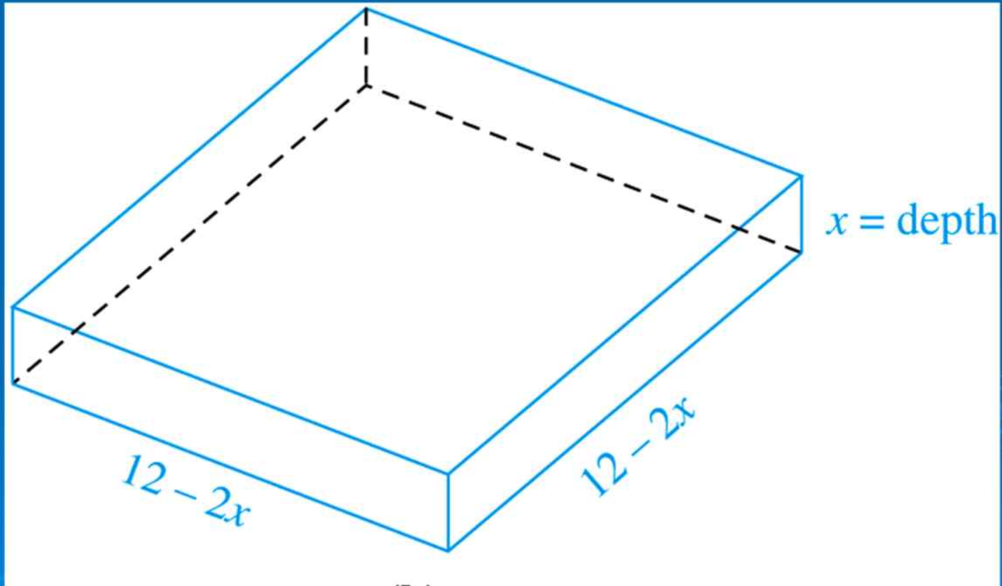

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Maximizing Volume

- 3. Volume is to be maximized so V = lwh = $x(12 - 2x)(12 - 2x) = 4x^3 - 48x^2 + 144x$
- 4. Domain: Volume can't be negative; neither can length, width, or height. What is the largest x that can be put into the equation without causing it to go negative?
- 6 Domain [0,6]

Maximizing Volume

- 6. Find the critical points and evaluate
- $V'(x) = 12x^2 - 96x + 144$
- $0 = 12(x2 - 8x + 12)$
- $0 = 12(x - 2)(x - 6)$
- $x = 2, 6$
- 7. Evaluate

When x = 2 in. the maximum volume is 128 in.^{3}

x	$V(x)$	
0	0	
2	128	← **Maximum**
6	0	

6.3 Business Applications: Lot Size

- A manufacturer must determine:
- What production level will result in minimum production and storage costs
- A purchaser must decide:
- How much to order to minimize reordering and storage costs

Suppose

- A company makes a constant number of units per year and that product can be made in several batches per year. The company can make:
- 1 large batch a year: minimum set up costs, but maximum storage costs
- Many small batches a year: maximum set up costs, but minimum storage costs
- The ideal combination is called economic lot size

Economic Lot Size

- Let:
- q = number of units in a batch
- k = cost of storing 1 unit for 1 year
- f = fixed setup costs
- g = cost of manufacturing 1 unit
- M = total number of units produced annually

Economic Lot Size

- The total cost of producing M units in batches of size q is:

$$T(q) = \frac{fM}{q} + gM + \frac{kq}{2}$$

$$T'(q) = \frac{-fM}{q^2} + \frac{k}{2}$$

- Setting the derivative = 0 gives $q = \sqrt{\frac{2fM}{k}}$

Economic Lot Size Example

- A paint company has a steady demand for 24,500 cans of automobile primer. It costs $2 to store one can of paint for a year and $500 in set up costs.
- How many cans of primer should be produced
- Find the number of batches per year to make to minimize total production costs.

Economic Lot Size

- Let:
- q = number of units in a batch
- k = cost of storing 1 unit for 1 year
- f = fixed setup costs
- g = cost of manufacturing 1 unit
- M = total number of units annually
- $k = 2$, $M = 24{,}500$, $f = 500$

Solution

- Using our critical number equation $q = \sqrt{\frac{2fM}{k}}$

$$q = \sqrt{\frac{2(500)(24,500)}{2}} = 3500$$

- The company should make 3500 cans per batch to minimize costs. The number of batches per year = $\frac{M}{q} = \frac{24,500}{3500} = 7$

Economic Order Quantity

- Let:
- q = number of units to order each time
- k = cost of storing 1 unit for 1 year
- f = fixed costs to place an order
- M = total number of units needed annually

Order Quantity

- A large pharmacy has an annual need for 480 units of a certain antibiotic. It costs $3 to store 1 unit for 1 year. The fixed cost of placing an order is $31.
- Find the number of units to order each time
- Find how many times a year the antibiotics should be ordered

Economic Order Quantity

- Let:
- q = number of units to order each time
- k = cost of storing 1 unit for 1 year
- f = fixed costs to place an order
- M = total number of units needed annually
- $k = 3$, $M = 480$, $f = 31$

Solution

- Using our critical number equation $q = \sqrt{\frac{2fM}{k}}$

$$q = \sqrt{\frac{2(31)(480)}{3}} \approx 99.6$$

- The company should order 100 units per order to minimize costs. The number of times per year = $\frac{M}{q} = \frac{480}{100} = 4.8$

Elasticity of Demand

- Changes in price affects demand
- Changes in price varies with different items (luxury items are more sensitive than essentials)
- One way to measure the sensitivity of demand in price is by the relative change—the ratio of percent change in demand to percent change in price. Elasticity of Demand is the measure

Elasticity of Demand

ELASTICITY OF DEMAND

Let $q = f(p)$, where q is demand at a price p. The **elasticity of demand** is

$$\boldsymbol{E = -\frac{p}{q} \cdot \frac{dq}{dp}}.$$

Demand is inelastic if $E < 1$.

Demand is elastic if $E > 1$.

Demand has unit elasticity if $E = 1$.

Example

- The demand for distilled spirits is measured by $q = f(p) = -.00375p + 7.87$, where p is the retail price of a case of liquor in dollars per case and
- q is the average number of cases purchased per year by a consumer.
- Calculate the elasticity of demand when p = \$118.30 per case

Example Con't

- $q = f(p) = -.00375p + 7.87$
- $dq/dp = -0.00375$

$$E = -\frac{p}{q} \bullet \frac{dq}{dp}$$

$$= -\frac{p}{-0.00375p + 7.87}(-.00375)$$

$$= -\frac{118.30}{-0.00375(118.30) + 7.87}(-.00375) \approx 0.0597$$

- Since E < 1, a percentage change in price will result in a smaller percentage change in demand, so an increase in price will increase revenue