Chapter 3: The Derivative

JMerrill, 2009

Review – Average Rate of Change

- Find the average rate of change for the function $f(x) = \frac{1}{x}$ from 1 to 5

$$-\frac{1}{5}$$

Review – Tangent Lines

- Find the tangent to the curve $f(x) = 3x^2 - 2$ at $x = 1$

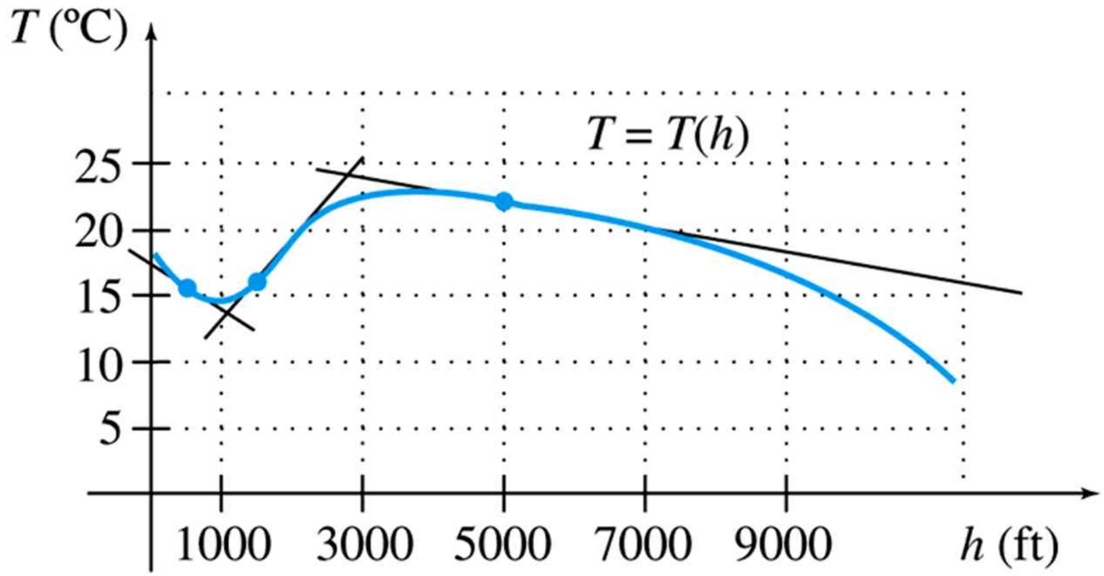
- $y = 6x - 5$

3.5 – Graphical Differentiation

- Given a graph of a cost function, how can we find the graph of the marginal cost function?
- Sometimes, all we have is a graph so it’s easier to find the derivative graphically

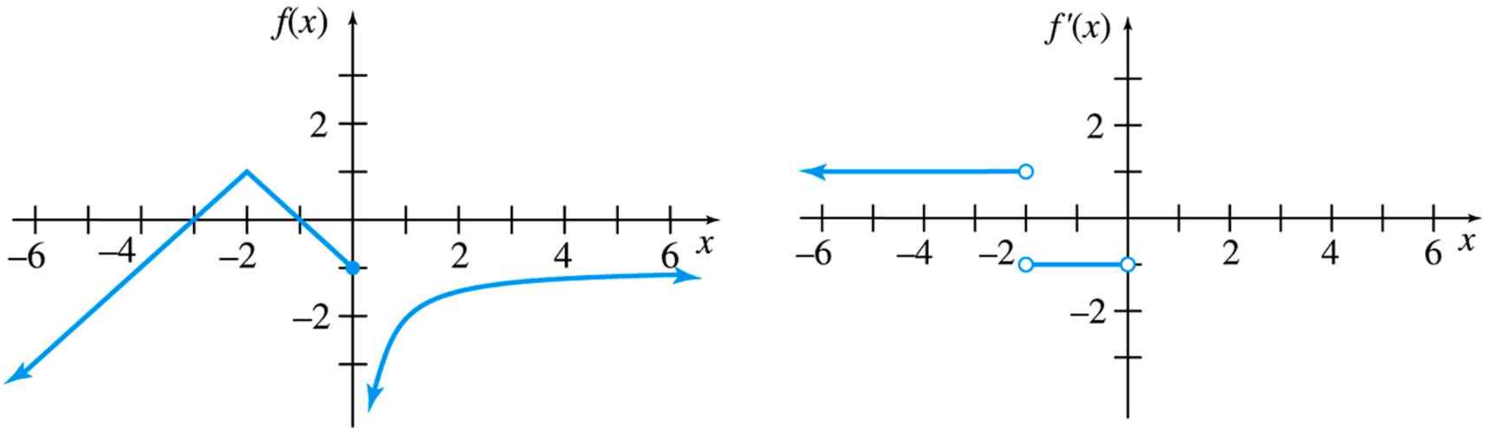
Graphing the Derivative

- When graphing the derivative, you are graphing the slope of the original function.



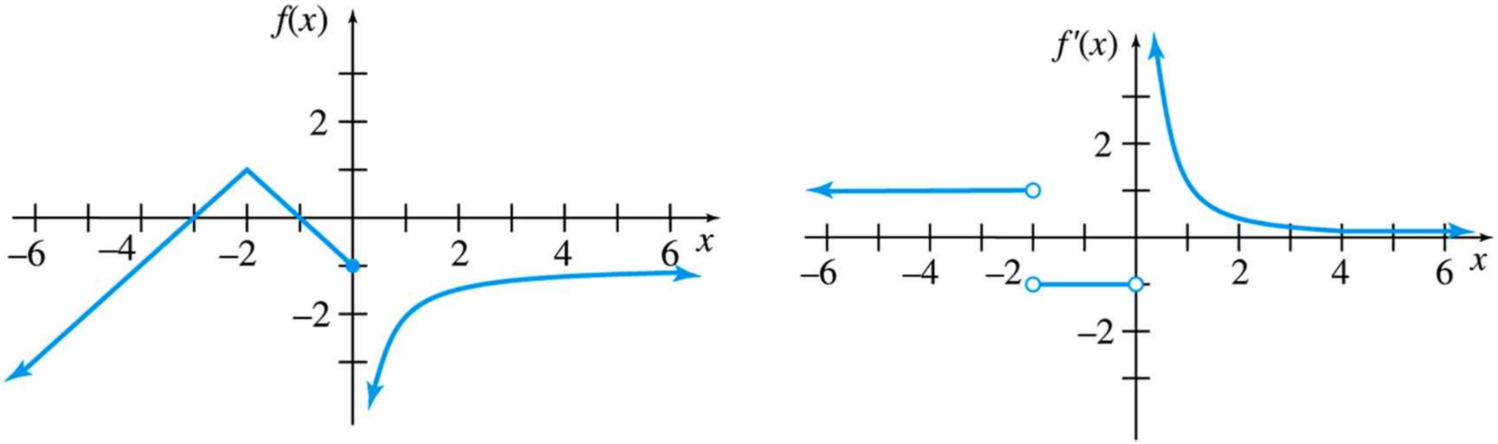
Graphing the Derivative

- When x < -2, the slope is 1
- When -2 < x < 0, the slope is -1
- At x = -2 and x = 0 the derivative does not exist—why?



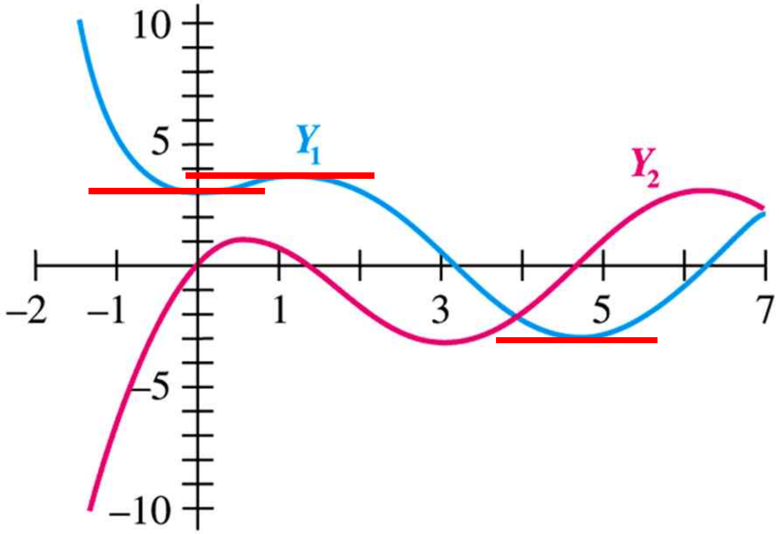
Graphing the Derivative

- For x > 0, the derivative is positive—estimated to be a slope of 1 at x = 1
- As x approaches 0 from the right, the derivative becomes larger
- As x approaches infinity, the derivative approaches 0.



Graphing

- Which is the f(x) and which is f'(x)?
- The derivative is 0 (crosses the x-axis) wherever there is a horizontal tangent
- Y1 = f(x)
- Y2 = f'(x)



Chapter 4 – Calculating the Derivative

- 4.1
- Techniques for Finding the Derivative

The Derivative

- The process of finding the derivative is changing. But the interpretation will not change—it is still the taking the limit as h approaches 0.

Notation

NOTATIONS FOR THE DERIVATIVE

The derivative of $y = f(x)$ may be written in any of the following ways:

$$f'(x), \qquad \frac{dy}{dx}, \qquad \frac{d}{dx}[f(x)], \qquad \text{or} \qquad D_x[f(x)].$$

Constant Rule

CONSTANT RULE

If $f(x) = k$, where k is any real number, then

$$f'(x) = 0.$$

(The derivative of a constant is 0.)

If f(x) = 4, then f'(x) = 0

If f(x) = π, then f'(x) = 0

Power Rule

POWER RULE

If $f(x) = x^n$ for any real number n, then

$$f'(x) = nx^{n-1}.$$

(The derivative of $f(x) = x^n$ is found by multiplying by the exponent n and decreasing the exponent on x by 1.)

Power Rule – Examples

- If $f(x) = x^6$, find $D_x y$
- $D_x y = 6x^{6-1} = 6x^5$
- If $f(x) = x, \text{find} \dfrac{dy}{dx}$

$$\frac{dy}{dx} = 1x^0 = 1$$

- If $y = \dfrac{1}{x^3} \text{find} \dfrac{dy}{dx}$

$\dfrac{1}{x^3}$ must be rewritten

x^{-3}

$$\frac{dy}{dx} = -3x^{-4} = \frac{-3}{x^4}$$

Power Rule Examples

- **Example 1:** Given $f(x) = 3x^2$, find $f'(x)$.
- $f'(x) = 6x$
- **Example 2:** Find the first derivative given $f(x) = 8x$.
- $8x^0 = 8$

Sum or Difference Rule

SUM OR DIFFERENCE RULE

If $f(x) = u(x) \pm v(x)$, and if $u'(x)$ and $v'(x)$ exist, then

$$\boldsymbol{f'(x) = u'(x) \pm v'(x)}.$$

(The derivative of a sum or difference of functions is the sum or difference of the derivatives.)

Sum/Difference Examples

- The Sum/Difference rule can be used on each term in a polynomial to find the first derivative.
- Find f'(x), given $f(x) = 5x^4 – 2x^3 – 5x^2 + 8x + 11$
- $f'(x) = 20x^3 – 6x^2 – 10x + 8$
- The derivative of a constant is 0 because 11 is the same as $11x^0$, which is $(0)11x^{-1}$

Sum/Difference Examples

- Find p'(t) given $p(t) = 12t^4 - 6\sqrt{t} = \dfrac{5}{t}$
- Rewrite p(t): $p(t) = 12t^4 - 6t^{\frac{1}{2}} + 5t^{-1}$

$$p'(t) = 48t^3 - 3t^{-\frac{1}{2}} - 5t^{-2}$$

$$p'(t) = 48t^3 - \frac{3}{\sqrt{t}} - \frac{5}{t^2}$$

Applications

- Marginal variables can be cost, revenue, and/or profit. Marginal refers to rates of change.
- Since the derivative gives the rate of change of a function, we find the derivative.

Application Example

- The total cost in hundreds of dollars to produce x thousand barrels of a beverage is given by
- $C(x) = 4x^2 + 100x + 500$
- Find the marginal cost for $x = 5$

- $C'(x) = 8x + 100$; $C'(5) = 140$

Example Continued

- After 5,000 barrels have been produced, the cost to produce 1,000 more barrels will be approximately $14,000
- The actual cost will be C(6) – C(5): 144 or $14,400

The Demand Function

- The Demand Function, defined by $p = D(q)$, related the number of q units of an item that consumers are will to purchase at the price, p.
- The total revenue $R(q)$ is related to price per unit and the amount demanded
- The total revenue is $R(q) = qp = qD(q)$

Demand Function Example

- The demand function for a certain product is given by $p = \dfrac{50{,}000 - q}{25{,}000}$
- Find the marginal revenue when q = 10,000 units and p is in dollars.

Demand Function Example

The revenue function is R(q) = qp

$$p = \frac{50{,}000 - q}{25{,}000}$$

$$R(q) = qp$$

$$= q\left(\frac{50{,}000 - q}{25{,}000}\right)$$

$$= \frac{50{,}000q - q^2}{25{,}000}$$

$$= 2q - \frac{1}{25{,}000}q^2$$

Example

- The marginal revenue is

$$2q - \frac{1}{25,000}q^2$$

$$R'(q) = 2 - \frac{2}{25,000}q$$

$$\text{For } q = 10,000$$

$$R'(10,000) = 2 - \frac{2}{25,000}(10,000) = 1.2$$

\$1.20 per unit

4.2

- Derivatives of Products and Quotients

Product Rule

PRODUCT RULE

If $f(x) = u(x) \cdot v(x)$, and if $u'(x)$ and $v'(x)$ both exist, then

$$\boldsymbol{f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x).}$$

(The derivative of a product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the first.)

Product Rule - Example

- Let $f(x) = (2x + 3)(3x^2)$. Find $f'(x)$

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x).$$

- $=(2x + 3)(6x) + (3x^2)(2)$
- $=12x^2 + 18x + 6x^2 = 18x^2 + 18x$

Power Rule

- Find f'(x) given that $f(x) = \left(\sqrt{x} + 3\right)\left(x^2 - 5x\right)$

$$\left(x^{\frac{1}{2}} + 3\right)(2x - 5) + \left(x^2 - 5x\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$\frac{5}{2}x^{\frac{3}{2}} + 6x - \frac{15}{2}x^{\frac{1}{2}} - 15$$

Quotient Rule

QUOTIENT RULE

If $f(x) = u(x)/v(x)$, if all indicated derivatives exist, and if $v(x) \neq 0$, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

(The derivative of a quotient is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.)

Quotient Rule Example

- Find f'(x) if $f(x) = \dfrac{2x-1}{4x+3}$

$$f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2}.$$

$$= \frac{(4x+3)(2)-(2x-1)(4)}{(4x+3)^2}$$

$$= \frac{10}{(4x+3)^2}$$

Product & Quotient Rules

- Find $D_x\left[\dfrac{(3-4x)(5x+1)}{7x-9}\right]$

$$\frac{(7x-9)D_x\left[(3-4x)(5x+1)\right]-\left[(3-4x)(5x+1)\right]D_x(7x-9)}{(7x-9)^2}$$

$$\frac{(7x-9)\left[(3-4x)(5)+(5x+1)(-4)\right]-(3+11x-20x^2)(7)}{(7x-9)^2}$$

$$\frac{-140x^2+360x-120}{(7x-9)^2}$$