Chapter 7: Integration

JMerrill, 2009

7.1 - Antiderivatives

- We have been solving situations dealing with total amounts of quantities
- Derivatives deal with the rate of change of those quantities
- Since it’s not always possible to find functions that deal with the total amount, we need to be able to find the rate of change of a given quantity
- Antidifferentiation is needed in this case

7.1 - Antiderivatives

ANTIDERIVATIVE

If $F'(x) = f(x)$, then $F(x)$ is an **antiderivative** of $f(x)$.

- If $F(x) = 10x$, then $F'(x) = 10$. $F(x)$ is the antiderivative of $f(x) = 10$
- If $F(x) = x^2$, $F'(x) = 2x$. $F(x)$ is the antiderivative of $f(x) = 2x$

7.1 - Antiderivatives

- Find the antiderivative of $f(x) = 5x^4$
- Work backwards (from finding the derivative)
- The antiderivative of $f(x)=F'(x)$ is x^5

7.1 - Antiderivatives

- In the example we just did, we know that $F(x) = x^2$ is not the only function whose derivative is $f(x) = 2x$
- $G(x) = x^2 + 2$ has $2x$ as the derivative
- $H(x) = x^2 - 7$ has $2x$ as the derivative

- For any real number, C, the function $F(x)=x^2 + C$ has $f(x)$ as an antiderivative

7.1 - Antiderivatives

- There is a whole family of functions having 2x as an antiderivative
- This family differs only by a constant

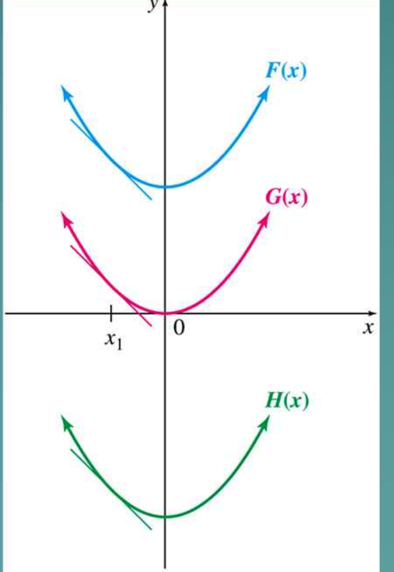
If $F(x)$ and $G(x)$ are both antiderivatives of a function $f(x)$ on an interval, then there is a constant C such that

$$F(x) - G(x) = C.$$

(Two antiderivatives of a function can differ only by a constant.) The arbitrary real number C is called an integration constant.

7.1 - Antiderivatives

- Since the functions
- $G(x) = x^2$ $F(x) = x^2 + 2$ $H(x) = x^2 - 7$ differ only by a constant, the slope of the tangent line remains the same



Slopes of the tangent lines at $x = x_1$ are the same.

The family of antiderivatives can be represented by $F(x) + C$

7.1 - Antiderivatives

- The family of all antiderivaties of f is indicated by

$$\int f(x)dx$$

Integral sign **Integrand**

This is called the indefinite integral and is the most general antiderivative of f

7.1 - Antiderivatives

INDEFINITE INTEGRAL

If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C,$$

for any real number C.

Example

- Using this new notation, $\int 2ax\, dx = x^2 + C$ the dx means the integral of f(x) with respect to x
- If we write $\int 2ax\, dx = a(2x)dx = ax^2 + C$ a gets treated as a constant and x as the variable
- If we write $\int 2ax\, da = a^2x + C = xa^2 + C$ x gets treated as the constant

Finding the Antiderivative

- Finding the antiderivative is the reverse of finding the derivative. Therefore, the rules for derivatives leads to a rule for antiderivatives
- Example: $$\frac{d}{dx}x^5 = 5x^4$$
- So $$\int 5x^4 dx = x^5 + C$$

Rules for Antiderivatives

- Power Rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$for\ any\ real\ number\ n \neq -1$$

$$(add\ 1\ to\ the\ exponent\ and\ divide\ by\ that\ number)$$

You can always check your answers by taking the derivative!

- Ex: $\int t^3 dt = \frac{t^{3+1}}{3+1} = \frac{t^4}{4} + C$

- Ex: $\int \frac{1}{t^2} dt = \int t^{-2} dt = \frac{t^{-1}}{-1} + C = \frac{-1}{t} + C$

You Do

- 1. $\int \sqrt{u}\ du$ $\qquad \frac{2}{3}u^{\frac{3}{2}} + C$

- 2. $\int dx$ $\qquad x + C$

Rules for Finding Antiderivatives

- Constant Multiple and Sum/Difference:

$$\int k \bullet f(x)dx = k\int f(x)dx$$

for any real number k

$$\int \left[f(x) \pm g(x) \right] dx = \int f(x) \pm \int g(x)dx$$

Examples

$$\int 2v^3 dv$$

$$2\int v^3 dv = 2\left(\frac{v^4}{4}\right) + C = \frac{v^4}{2} + C$$

- You do:

$$\int \frac{12}{z^5} dz \qquad \frac{-3}{z^4} + C$$

$$\int \left(3z^2 - 4z + 5\right) dz \qquad z^3 - 2z^2 + 5z + C$$

Example

$$\int \frac{x^2+1}{\sqrt{x}} = \int\left(\frac{x^2}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)dx$$

First, rewrite the integrand

$$= \int\left(\frac{x^2}{x^{\frac{1}{2}}}+\frac{1}{x^{\frac{1}{2}}}\right)dx = \int\left(x^{\frac{3}{2}}+x^{\frac{-1}{2}}\right)dx$$

Now that we have rewritten the integral, we can find the antiderivative

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+C = \frac{2}{5}x^{\frac{5}{2}}+2x^{\frac{1}{2}}+C$$

Recall

- Previous learning:
- If $f(x) = e^x$ then $f'(x) = e^x$
- If $f(x) = a^x$ then $f'(x) = (\ln a)a^x$
- If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$
- If $f(x) = a^{kx}$ then $f'(x) = k(\ln a)a^{kx}$

- This leads to the following formulas:

Indefinite Integrals of Exponential Functions

$$\int e^x dx = e^x + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C, k \neq 0$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int a^{kx} dx = \frac{a^{kx}}{k(\ln a)} + C, k \neq 0$$

This comes from the chart on P. 434

Examples

$$\int 9e^{t}\,dt = 9\int e^{t}\,dt = 9e^{t} + C$$

$$\int e^{9t}\,dt = \frac{e^{9t}}{9} + C$$

$$\int 3e^{\frac{5}{4}u}\,du = 3\left(\frac{e^{\frac{5}{4}u}}{\frac{5}{4}}\right) + C = 3\left(\frac{4}{5}\right)e^{\frac{5}{4}u} + C = \frac{12}{5}e^{\frac{5}{4}u} + C$$

You Do

$$\int 2^{-5x}\,dx$$

$$= \frac{-2^{-5x}}{5(\ln 2)} + C$$

Indefinite Integral of x^{-1}

$$\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$$

- Note: if x takes on a negative value, then lnx will be undefined. The absolute value sign keeps that from happening.

Example

$$\int \frac{4}{x}dx = 4\int \frac{1}{x}dx = 4\ln|x| + C$$

You Do:

$$\int \left(\frac{-5}{x} + e^{-2x} \right) dx$$

$$-5\ln|x| - \frac{1}{2}e^{-2x} + C$$

Application - Cost

- Suppose a publishing company has found that the marginal cost at a level of production of of x thousand books is given by $C'(x) = \frac{50}{\sqrt{x}}$ and that the fixed cost

  (before any book is published) is $25,000. Find the cost function.

Solution

$$C'(x) = \frac{50}{\sqrt{x}}$$

First, rewrite the function.

$$C'(x) = 50x^{\frac{-1}{2}}$$

$$\int 50x^{\frac{-1}{2}}dx = 50\int x^{\frac{-1}{2}}dx = 50\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right) + K$$

$$= 50\left(2x^{\frac{1}{2}}\right) + k = 100x^{\frac{1}{2}} + k$$

Before any books are produced the fixed cost is \$25,000—so C(0)=25,000

$$C(x) = 100x^{\frac{1}{2}} + k$$

$$25{,}000 = 100(0) + k$$

$$25{,}000 = k$$

$$\boxed{C(x) = 100x^{\frac{1}{2}} + 25{,}000}$$

Application - Demand

- Suppose the marginal revenue from a product is given by $400e^{-0.1q} + 8$. a) Find the revenue function.
- $R'(q) = 400e^{-0.1q} + 8$

Set R and q = 0 to solve for C.

$$R(q) = \int (400e^{-0.1q} + 8)dq$$

$$= 400\frac{e^{-0.1q}}{-0.1} + 8q + C$$

$$= -4000e^{-0.1q} + 8q + C$$

$$0 = -4000e^{-0.1(0)} + 8(0) + C$$

$$4000 = C$$

- $R(q) = 400e^{-0.1q} + 8q + 4000$

Application - Demand

- B) Find the demand function.
- Recall that R = qp where p is the demand function
- $R = qp$
- $400e^{-0.1q} + 8q + 4000 = qp$
- $\dfrac{400e^{-0.1q} + 8q + 4000}{q} = p$

7.2 - Substitution

- In finding the antiderivative for some functions, many techniques fail
- Substitution can sometimes remedy this problem
- Substitution depends on the idea of a differential.
- If $u = f(x)$, then the differential of u, written du, is defined as $du = f'(x)dx$
- Example: If $u=2x^3 + 1$, then $du=6x^2\,dx$

Example

- $\int \left(2x^3+1\right)^4 6x^2dx$ looks like the chain rule and product rule.
- But using differentials and substitution we'll find the antiderivative

$$\int \left(2x^3+1\right)^4 6x^2dx = \int \overbrace{\left(2x^3+1\right)}^{u}{}^4 \overbrace{6x^2dx}^{du}$$

$$= \int u^4du$$

Example Con't

- Now use the power rule

$$\int u^4 du \quad = \frac{u^5}{5} + C$$

- Substitute $(2x^3 + 1)$ back in for u:

$$\int \left(2x^3+1\right)^4 6x^2 dx = \frac{\left(2x^3+1\right)^5}{5} + C$$

You Do

- Find $\int 6x\left(3x^2+4\right)^7 dx$

$$\int \left(\overset{u}{3x^2+4}\right)^7 \overset{du}{6x\,dx} \quad = \int u^7 du$$

$$\int u^7 du = \frac{u^8}{8} + C \;\; = \frac{\left(3x^2+4\right)^8}{8} + C$$

Choosing u

SUBSTITUTION METHOD

In general, for the types of problems we are concerned with, there are three cases. We choose u to be one of the following:

1. the quantity under a root or raised to a power;
2. the quantity in the denominator;
3. the exponent on e.

Remember that some integrands may need to be rearranged to fit one of these cases.

du

- We haven't needed the du in the past 2 problems, but that's not always the case. The du happened to have already appeared in the previous examples.
- Remember, du is the derivative of u.

$$\int \left(2x^3 + 1\right)^4 6x^2 dx \qquad \int \left(3x^2 + 4\right)^7 6x dx$$

Example

- Find $\int x^2\sqrt{x^3+1}dx$

- Let u = $x^3 + 1$, then du = $3x^2dx$
- There's an x^2 in the problem but no $3x^2$, so we need to multiply by 3
- Multiplying by 3 changes the problem, so we need to counteract that 3 by also multiplying by 1/3

Example

$$\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int 3x^2\sqrt{x^3+1}\,dx$$

$$= \frac{1}{3}\int \sqrt{x^3+1}\,\left(3x^2dx\right) = \frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3}\int u^{\frac{1}{2}}du$$

$$= \frac{1}{3}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C = \frac{2}{9}u^{\frac{3}{2}} + C$$

$$= \frac{2}{9}\left(x^3+1\right)^{\frac{3}{2}} + C$$

Example

- Find $\displaystyle\int \frac{(x+3)}{\left(x^2+6x\right)^2}dx$

- u = x² + 6x, so du = (2x + 6)

$$\int \frac{(x+3)}{\left(x^2+6x\right)^2}dx = \frac{1}{2}\int \frac{2(x+3)}{\left(x^2+6x\right)^2}dx = \frac{1}{2}\int \frac{du}{u^2}$$

$$= \frac{1}{2}\int u^{-2}du = \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + C = \frac{-1}{2u} + C$$

$$= \frac{-1}{2\left(x^2+6x\right)} + C$$

7.3-Area & The Definite Integral

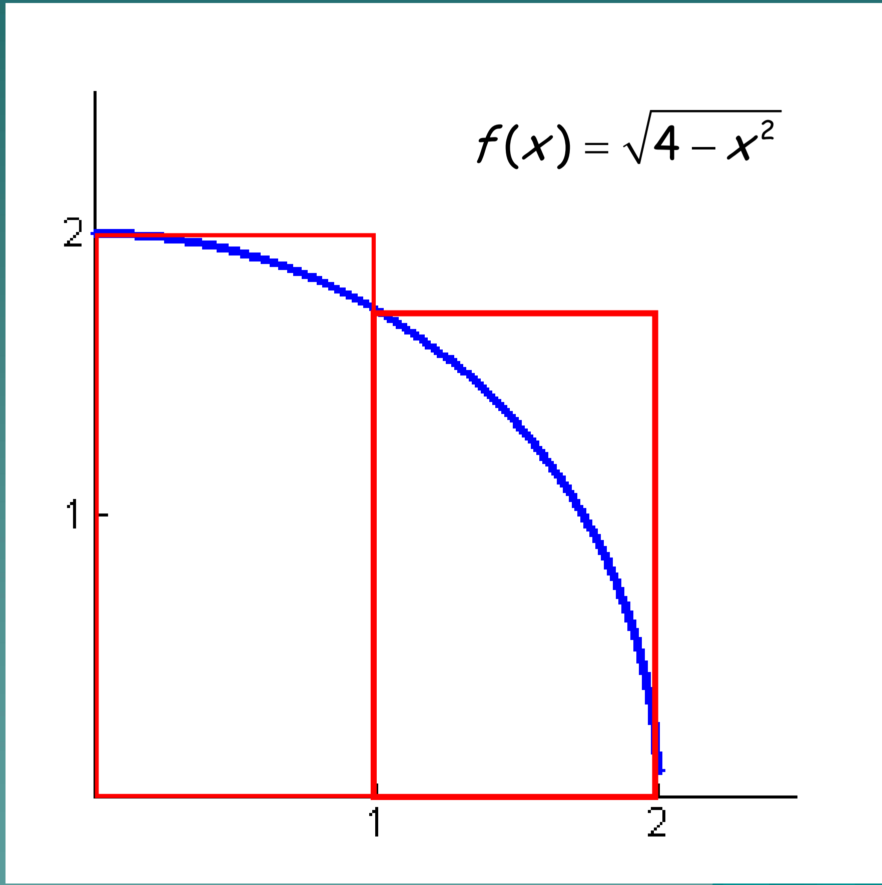
- We'll start with Archimedes! Yea!

Archimedes Method of Exhaustion

- To find the area of a regular geometric figure is easy. We simply plug the known parts into a formula that has already been established.
- But, we will be finding the area of regions of graphs—not standard geometric figures.
- Under certain conditions, the area of a region can be thought of as the sum of its parts.

Archimedes Method of Exhaustion

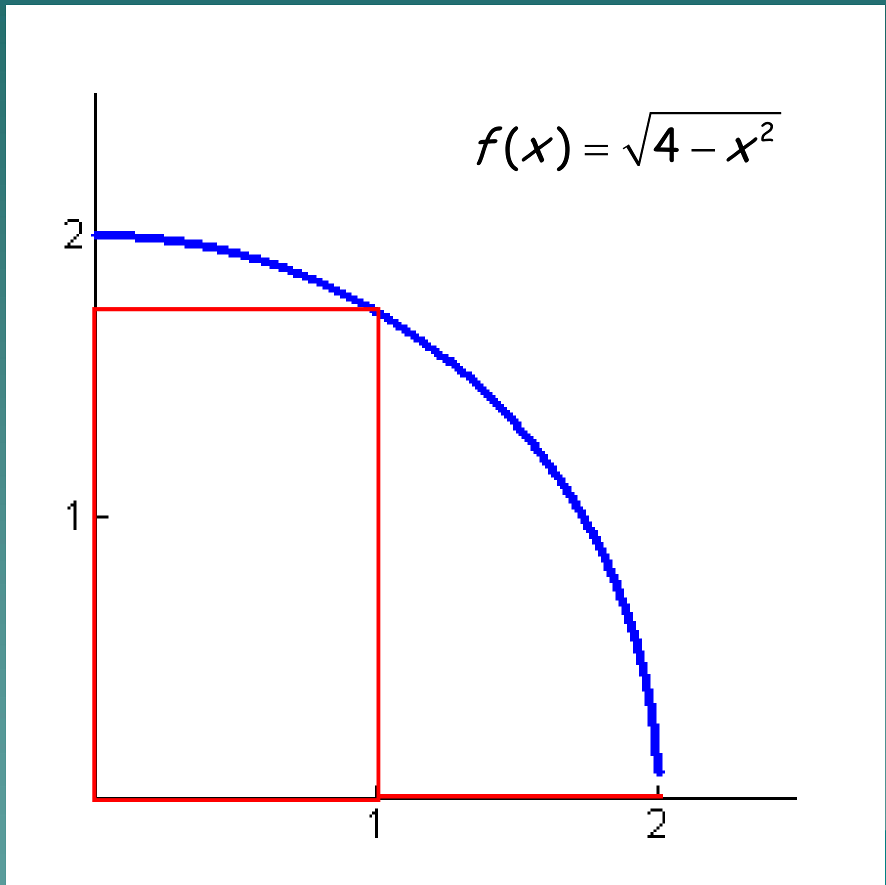
- A very rough approximation of this area can be found by using 2 inscribed rectangles.
- Using the left endpoints, the height of the left rectangle is f(0)=2. The height of the right rectangle is f(1)=√3
- $A=1(2)+1(\sqrt{3})=3.7321u^2$



Over estimate or under estimate?

Archimedes Method of Exhaustion

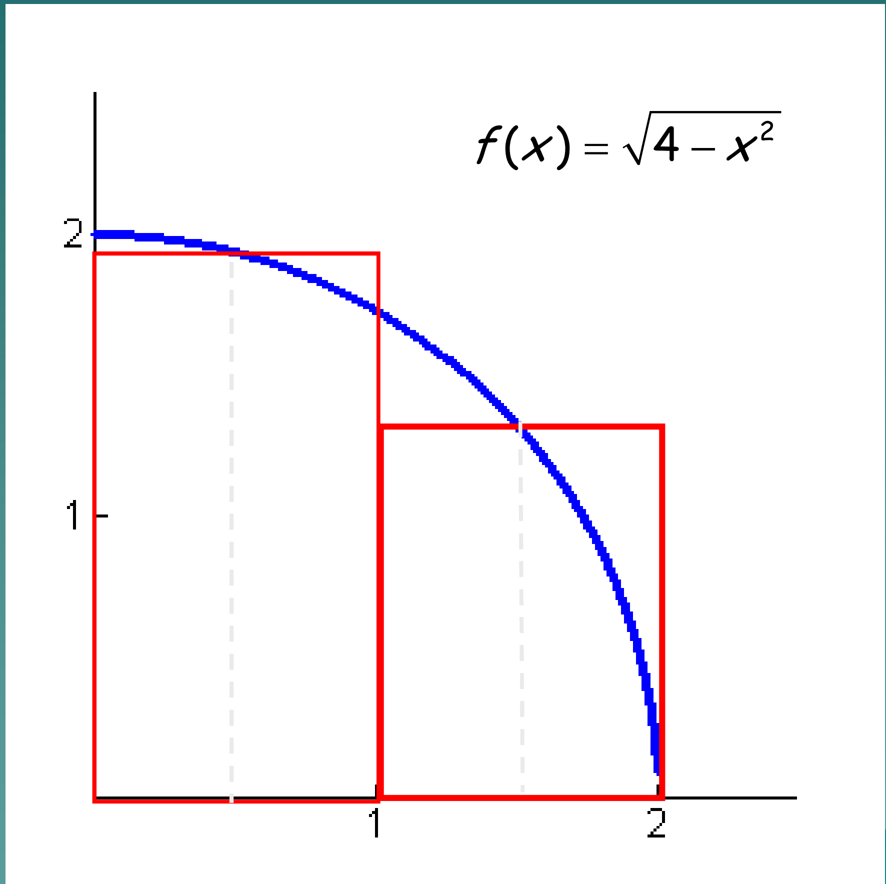
- We can also estimate using the right endpoints. The height of the left rectangle is f(1)=√3. The other height is f(2)=0.
- A=1(√3)+1(0)=1.7321u^2



Over estimate or under estimate?

Archimedes Method of Exhaustion

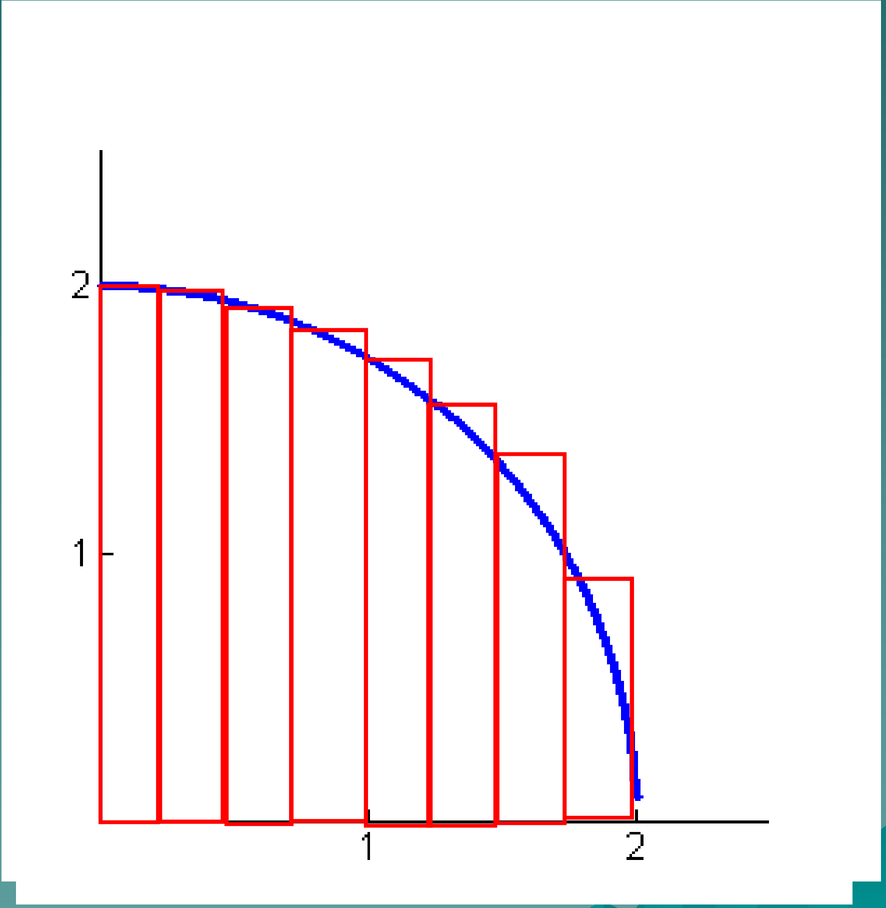
- We could average the 2 to get 2.7321 or use the midpoints of the rectangles:
- A=1(f(.5))+1(f(1.5))
- = √3.75+ √1.75 =3.2594u²



Better estimate?

Archimedes Method of Exhaustion

- To improve the approximation, we can divide the interval from x=0 to x=2 into more rectangles of equal width.
- The width is determined by $\frac{2-0}{n}$ with n being the number of equal parts.

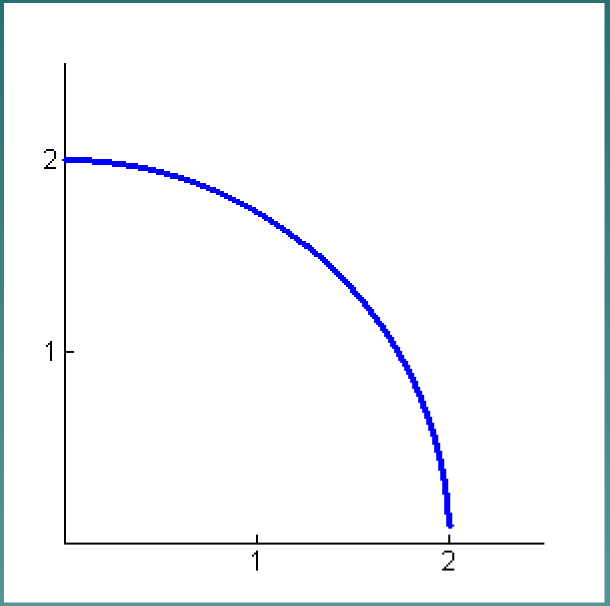


Area

- We know that this is a quarter of a circle and we know the formula for area of a circle is $A=\pi r^2$.
- $A=1/4\ \pi(2)^2$

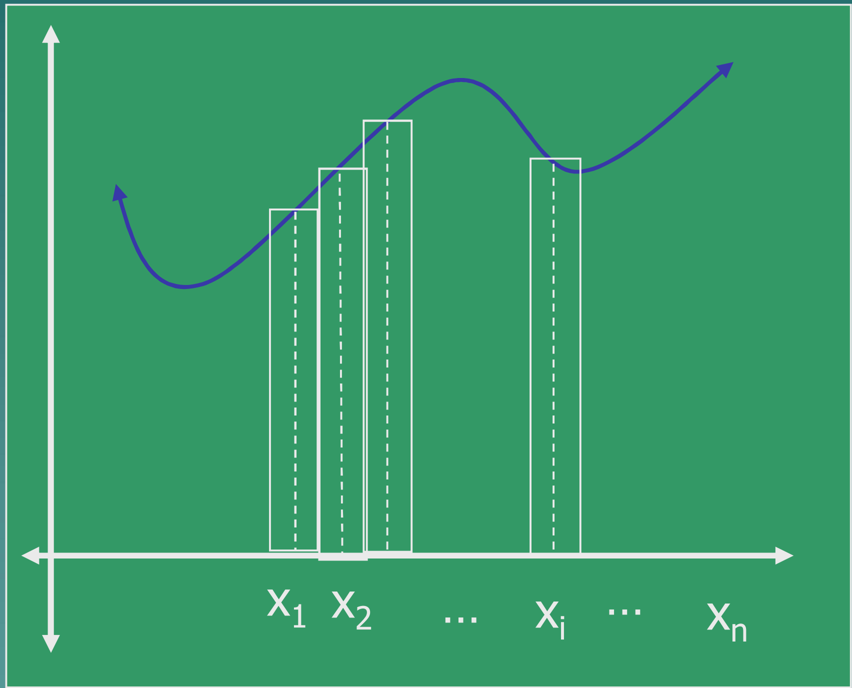
  $=3.1416\text{units}^2$

To develop a process that results in the exact area, begin by dividing the interval from *a* to *b* into *n* pieces of equal width.



Exact Area

- x_1 is an arbitrary point in the 1st rectangle, x_2 in the 2nd and so on.
- Δx represents the width of each rectangle
- Area of all n rectangles = $\sum_{i=1}^{n} f(x_i)\Delta x$



Exact Area

- The exact area is defined to be the sum of the limit (if the limit exists) as the number of rectangles increases without bound. The exact area =

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

The Definite Integral

- If *f* is defined on the interval [a,b], the definite integral of *f* from *a* to *b* is given by

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

provided the limit exists, where delta x = (b-a)/n and x_i is any value of x in the i^{th} interval.

- The interval can be approximated by

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

(The sum of areas of all the triangles!)

The Definite Integral

- Unlike the indefinite integral, which is a set of functions, the definite integral represents a *number*

Upper limit

$$\int_a^b f(x)dx$$

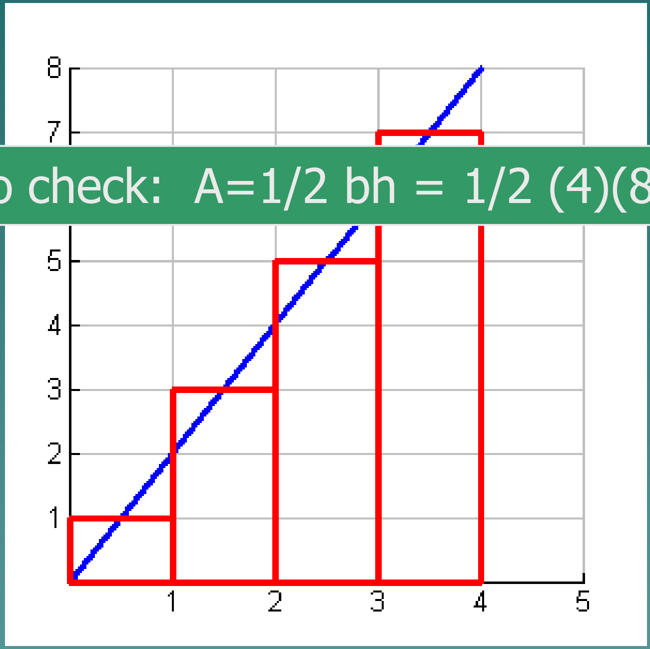
Lower limit

The Definite Integral

- The definite integral can be thought of as a mathematical process that gives the sum of an infinite number of individual parts. It represents the area only if the function involved is nonnegative ($f(x) \geq 0$) for every x-value in the interval [a,b].
- There are many other interpretations of the definite integral, but all involve the idea of approximation by sums.

Example

- Approximate $\int_0^4 2x\,dx$ the area of the region under the graph of f(x) = 2x above the x-axis, and between x=0 and x=4. Use 4 rectangles of equal width whose heights are the values of the function at the midpoint of each subinterval



To check: A=1/2 bh = 1/2 (4)(8)=16

$$\sum_{i=1}^{4} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$= f(.5)\Delta x + f(1.5)\Delta x + f(2.5)\Delta x + f(3.5)\Delta x$$

$$= 1(1) + 3(1) + 5(1) + 7(1)$$

$$= 16 units^2$$

Total Change in F(x)

- The total change in a quantity can be found from the function that gives the rate of change of the quantity, using the same methods used to approximate the area under the curve:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$