Antiderivatives

(7.4, 8.2, 10.1)

JMerrill, 2009

Review Info - Antiderivatives

- General solutions:

Variable of Integration

$$y = \int f(x)dx = F(x) + C$$

Integrand

Constant of Integration

Review

- Rewriting & Integrating – general solution

Original	Rewrite	Integrate	Simplify
$\int \frac{1}{x^3} dx$	$\int x^{-3} dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
$\int \sqrt{x}\ dx$	$\int x^{\frac{1}{2}} dx$	$\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$	$\frac{2}{3} x^{\frac{3}{2}} + C$

Particular Solutions

- To find a particular solution, you must have an initial condition
- Ex: Find the particular solution of $F'(x) = \frac{1}{x^2}$ that satisfies the condition F(1) = 0

$$F'(x) = \frac{1}{x^2} dx$$

$$= \int x^{-2} dx$$

$$= \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

$$F(x) = -\frac{1}{x} + 1$$

$$F(x) = -\frac{1}{x} + C$$

$$F(1) = -\frac{1}{1} + C$$

$$0 = -1 + C$$

$$1 = C$$

Indefinite & Definite Integrals

- Indefinite Integrals have the form:

$$\int f(x)dx$$

- Definite integrals have the form:

$$\int_a^b f(x)dx$$

7.4 The Fundamental Theorem of Calculus

- This theorem represents the relationship between antiderivatives and the definite integral

FUNDAMENTAL THEOREM OF CALCULUS

Let f be continuous on the interval $[a, b]$, and let F be *any* antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

Here’s How the Theorem Works

- First find the antiderivative, then find the definite integral

$$\int_1^2 4x^3dx = x^4\Big|_1^2 = 2^4 - 1^4 = 15$$

$$\int 4x^3dx = \frac{4x^4}{4} = x^4$$

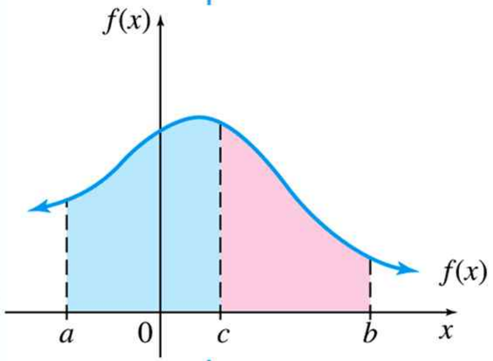
Properties of Definite Integrals

- The chart on P. 466:

PROPERTIES OF DEFINITE INTEGRALS

If all indicated definite integrals exist,

1. $\int_a^a f(x)\,dx = 0$;
2. $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$ for any real constant k (constant multiple of a function);
3. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (sum or difference of functions);
4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any real number c;
5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.



Example – Sum/Difference

- Find $\int_2^5 \left(6x^2 - 3x + 5\right)dx$

$$= 6\int_2^5 x^2 dx - 3\int_2^5 x dx + 5\int_2^5 dx$$

$$= 2x^3\Big|_2^5 - \frac{3}{2}x^2\Big|_2^5 + 5x\Big|_2^5$$

$$= 2\left(5^3 - 2^3\right) - \frac{3}{2}\left(5^2 - 2^2\right) + 5(5-2)$$

$$= 234 - \frac{63}{2} + 15 = \frac{435}{2}$$

$$6\int x^2 = 6\left(\frac{x^3}{3}\right) = 2x^3$$

$$3\int x dx = 3\left(\frac{x^2}{2}\right) = \frac{3}{2}x^2$$

$$5\int dx = 5(x) = 5x$$

Less Confusing Notation?

- Evaluate $\int_0^2 \left(2x^2 - 3x + 2\right)dx$

$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2$$

$$= \left(\frac{16}{3} - 6 + 4\right) - (0 - 0 + 0)$$

$$= \frac{10}{3}$$

Substitution - Review

- Evaluate $\int 3(3x-1)^4\,dx$

- Let u = 3x – 1; du = 3dx

$$\int \overbrace{(3x-1)^4}^{u^4}\,\overbrace{3dx}^{du} = \frac{u^5}{5} + C = \frac{(3x-1)^5}{5} + C$$

Substitution & The Definite Integral

- Evaluate $\int_0^5 x\sqrt{25-x^2}\,dx$
- Let u = 25 – x^2; du = -2xdx

$$= -\frac{1}{2}\int_0^5 \sqrt{25-x^2}\,(-2x\,dx)$$

$$= -\frac{1}{2}\int_0^5 \sqrt{u}\,du = -\frac{1}{2}\int_0^5 u^{\frac{1}{2}}\,du$$

$$= -\frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C \quad = -\frac{u^{\frac{3}{2}}}{3} + C$$

$$= \left[-\frac{(25-x^2)^{\frac{3}{2}}}{3}\right]_0^5$$

$$= \frac{0}{3} - \left(-\frac{(25)^{\frac{3}{2}}}{3}\right) = \frac{125}{3}$$

Area

- Find the area bounded by the curve of f(x) = $(x^2 - 4)$, the x-axis, and the vertical lines x = 0, x = 2

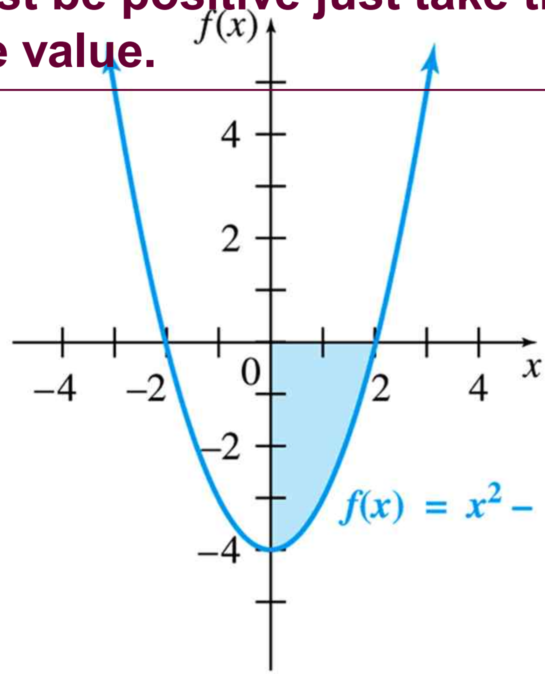
$$\int_0^2 \left(x^2 - 4\right)dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_0^2$$

$$= \left(\frac{8}{3} - 8\right) - (0 - 0) = -\frac{16}{3}$$

$$\frac{16}{3}$$

The answer is negative because the area is below the x-axis. Since area must be positive just take the absolute value.



Finding Area

FINDING AREA

In summary, to find the area bounded by $f(x)$, $x = a$, $x = b$, and the x-axis, use the following steps.

1. Sketch a graph.
2. Find any x-intercepts of $f(x)$ in $[a, b]$. These divide the total region into subregions.
3. The definite integral will be *positive* for subregions above the x-axis and *negative* for subregions below the x-axis. Use separate integrals to find the (positive) areas of the subregions.
4. The total area is the sum of the areas of all of the subregions.

Area – Last Example

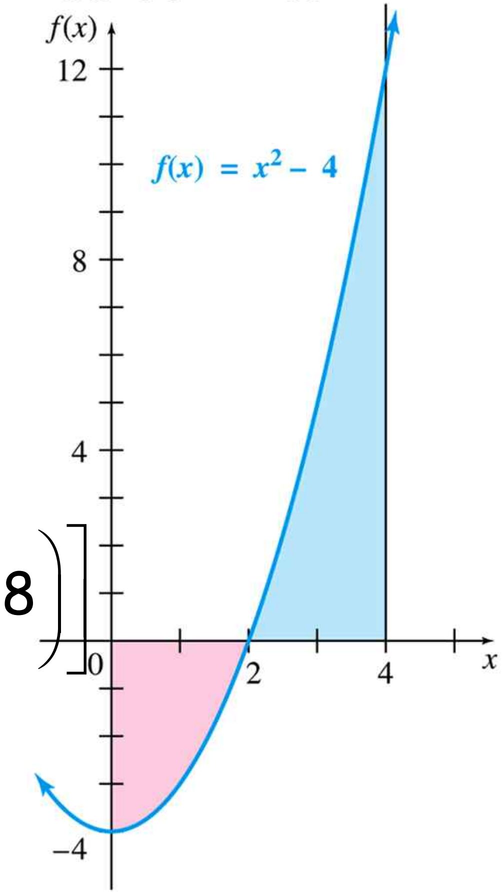
- Find the area between the x-axis and the graph of f(x) = x^2 – 4 from x = 0 to x = 4.

$$\int_0^2 \left(x^2-4\right)dx+\int_2^4 \left(x^2-4\right)dx$$

$$=\left|\left[\frac{1}{3}x^3-4x\right]_0^2\right|+\left[\frac{1}{3}x^3-4x\right]_2^4$$

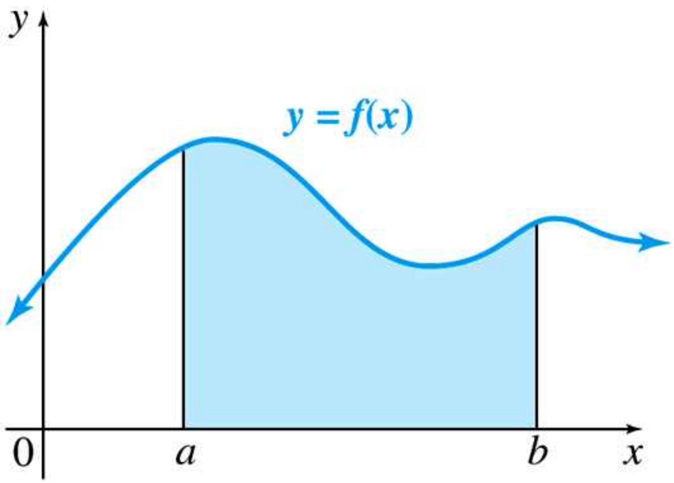
$$=\left|\left[\left(\frac{8}{3}-8\right)-(0-0)\right]\right|+\left[\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)\right]$$

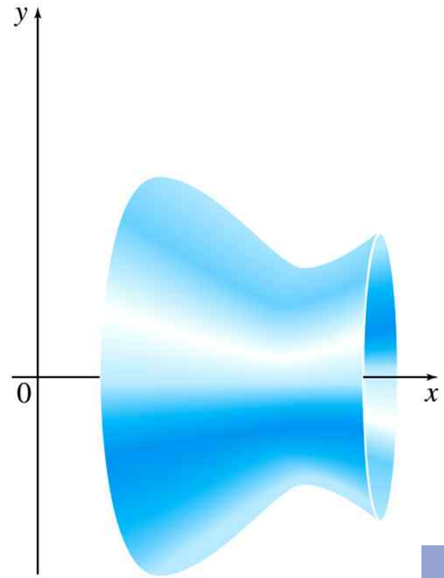
$$=16$$



8.2 Volume & Average Value

- We have used integrals to find the area of regions. If we rotate that region around the x-axis, the resulting figure is called a solid of revolution.





Volume of a Solid of Revolution

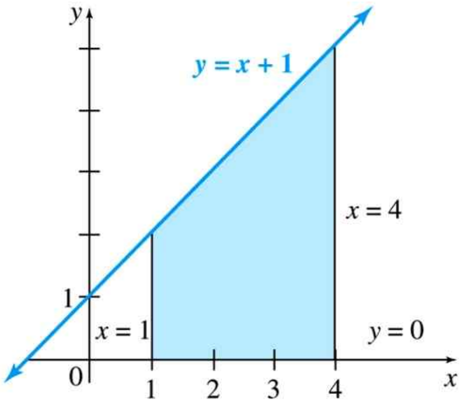
VOLUME OF A SOLID OF REVOLUTION

If $f(x)$ is nonnegative and R is the region between $f(x)$ and the x-axis from $x = a$ to $x = b$, the volume of the solid formed by rotating R about the x-axis is given by

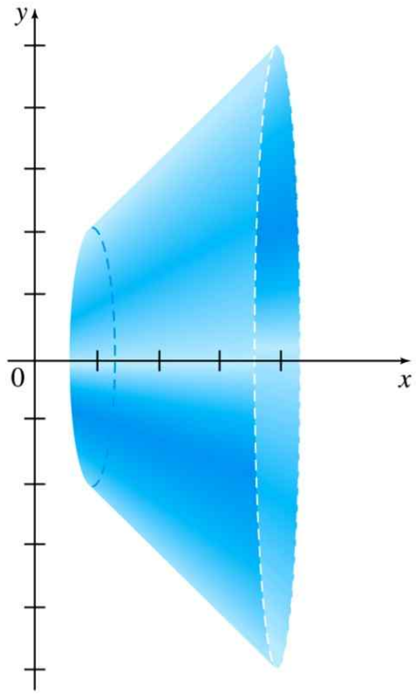
$$V = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi [f(x_i)]^2 \Delta x = \int_a^b \pi [f(x)]^2 \, dx.$$

Volume Example

- Find the volume of the solid of revolution formed by rotating about the x-axis the region bounded by $y = x + 1$, $y = 0$, $x = 1$, and $x = 4$.



(a)



(b)

Volume Example

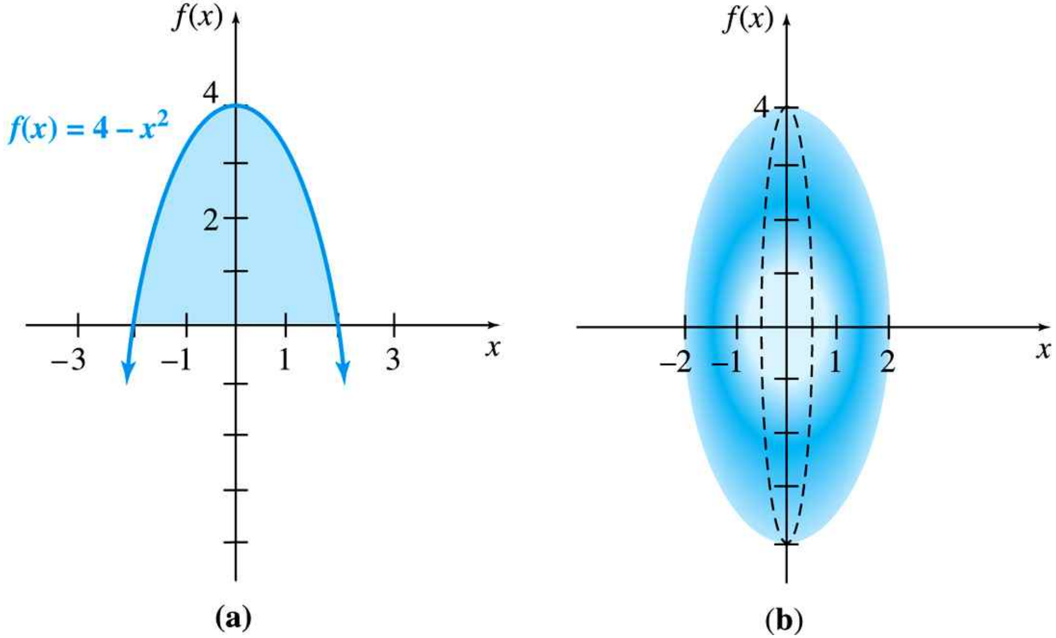
$$V = \int_1^4 \pi (x+1)^2 dx$$

$$= \pi \left[\frac{(x+1)^3}{3} \right]_1^4$$

$$= \frac{\pi}{3}\left(5^3 - 2^3\right) = \frac{117\pi}{3} = 39\pi$$

Volume Problem

- Find the volume of the solid of revolution formed by rotating about the x-axis the area bounded by f(x) = 4 – x^2 and the x-axis.



Volume Con't

$$V = \int_{-2}^{2} \pi \left(4 - x^2\right)^2 dx$$

$$= \int_{-2}^{2} \pi \left(16 - 8x^2 + x^4\right) dx$$

$$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^{2}$$

$$= \left(32 - \frac{64}{3} + \frac{32}{5}\right) - \left(-32 + \frac{64}{3} - \frac{32}{5}\right) = \frac{512\pi}{15}$$

Average Value

AVERAGE VALUE OF A FUNCTION

The **average value of a function** f on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx,$$

provided the indicated definite integral exists.

Average Price

- A stock analyst plots the price per share of a certain common stock as a function of time and finds that it can be approximated by the function
- $$S(t)=25 - 5e^{-.01t}$$
- where t is the time (in years) since the stock was purchased. Find the average price of the stock over the first 6 years.

Avg Price - Solution

- We are looking for the average over the first 6 years, so a = 0 and b = 6.

$$\frac{1}{6-0}\int_0^6 \left(25-5e^{-.01t}\right)dt$$

$$=\frac{1}{6}\left[25t-\frac{5}{-.01}e^{-.01t}\right]_0^6$$

$$=\frac{1}{6}\left(150+500e^{-.06}-500\right)$$

$$=20.147$$

The average price of the stock is about \$20.15

10.1 Differential Equations

- A differential equation is one that involves an unknown function $y = f(x)$ and a finite number of its derivatives. Solving the differential equation is used for forecasting interest rates.
- A solution of an equation is a number (usually).
- A solution of a differential equation is a function.

Differential Equations

GENERAL SOLUTION OF $\dfrac{dy}{dx} = f(x)$

The general solution of the differential equation $\dfrac{dy}{dx} = f(x)$ is

$$\boldsymbol{y = \int f(x)\,dx.}$$

Population Example

- The population, P, of a flock of birds, is growing exponentially so that $\frac{dP}{dx} = 20e^{0.05x}$, where x is time in years.

- Find P in terms of x if there were 20 birds in the flock initially.

Note: Notice the denominator has the same variable as the right side of the equation.

Population Cont

- Take the antiderivative of each side:

$$\frac{dP}{dx} = 20e^{0.05x}$$

$$P = \int 20e^{0.05x}dx$$

$$= \frac{20}{0.05}e^{0.05x} + C = 400e^{0.05x} + C$$

$$20 = 400e^{0} + C$$

$$-380 = C$$

$$P = 400e^{0.05x} - 380$$

- This is an initial value problem. At time 0, we had 20 birds.

One More Initial Value Problem

- Find the particular solution of $\frac{dy}{dx} - 2x = 5$ when y = 2, x = -1

$$\frac{dy}{dx} = 2x + 5$$

Note: Notice the denominator has the same variable as the right side of the equation.

$$\int \frac{dy}{dx} = \int 2x + 5dx$$

$$y = \frac{2x^2}{2} + 5x + C = x^2 + 5x + C$$

$$2 = (-1)^2 + 5(-1) + C$$

$$6 = C$$

$$y = x^2 + 5x + 6$$

Separation of Variables

- Not all differential equations can be solved this easily.
- If interest is compounded continuously then the money grows at a rate proportional to the amount of money present and would be modeled by $\frac{dA}{dt} = kA$

Note: Notice the denominator does not have the same variable as the right side of the equation.

Separation of Variables

- In general terms think of $\frac{dy}{dx} = \frac{f(x)}{g(y)}$
- This of dy/dx as the fraction dy over dx (which is totally incorrect, but it works!)
- In this case, we have to separate the variables

$$\int g(y)dy = \int f(x)dx$$

$$G(y) = F(x) + C$$

(Get all the y's on one side and all the x's on the other)

Example

- Find the general solution of $y\dfrac{dy}{dx} = x^2$
- Multiply both sides by dx to get $y\,dy = x^2 dx$

$$\int y\,dy = \int x^2 dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + C$$

$$y^2 = \frac{2}{3}x^3 + 2C$$

Lab 4 – Due Next Time on Exam Day

- 1. #34, P471
- 2. #59, P440
- 3. #22, P471
- 4. #45, P439
- 5. #11, P439
- 6. #13, P471
- 7. #27, P439
- 8. #17, P522

9. #25, P523
10. #35, P523
11. #3, P629
12. #7, P629
13. #19, P630
14. #27, P630
15. #43, P472