Chapter 4. Unsteady-State Conduction

4-1 INTODUCTION

If a solid body is suddenly subjected to a change in environment some time must elapse before an equilibrium temperature condition will prevail in the body. During this transient period the temperature change and the analysis must take into account changes in the internal energy of the body with time. Unsteady-state heat transfer analysis is obviously of significant practical interest because of the large number of heating and cooling processes which are time-dependent.

To analyze a transient heat transfer problem, we could proceed by solving the general heat-conduction equation by the separation of variables method, similar to the analytical treatment used for the two-dimensional steady-state problem.

Consider the infinite plate of thickness L, initially at a uniform temperature T_i, at time zero. The surface is suddenly lowered to and subsequent times maintained at a constant temperature T_1. It is desired to find the temperature at any location in the plate as a function of time for $t>0$, i.e, $T(x,t)$.

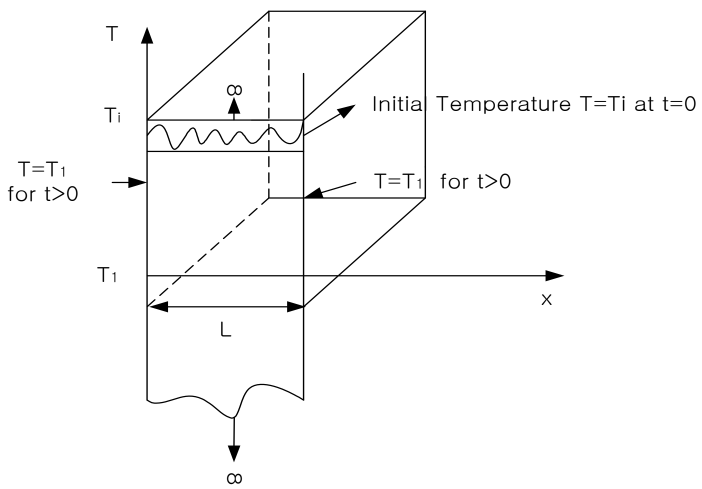


Figure. The infinite plate, initially at uniform temperature, subjected at time zero to sudden cooling of surface.

The governing equation with constant thermo-physical properties and no internal heat sources is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

B.C. $T = T_1$ at $x=0$, $t>0$ **or** $T(0,t) = T_1$

$T = T_1$ at $x=L$, $t>0$ **or** $T(L,t) = T_1$

I.C. $T = T_i$ at $0 \le x \le L$, $t=0$ **or** $T(x,0) = Ti$

It is desirable to work in terms of a temperature difference variable $\Theta = T - T_1$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \quad \text{(Heat diffusion equation)} \qquad -①$$

Boundary and Initial conditions

$\Theta(0,t)=0$ at $x=0$, $t>0$ (a)

$\Theta(L,t)=0$ at $x=L$, $t>0$ (b)

$\Theta(x,0)=\Theta_i$ at $0 \le x \le L$, $t=0$ (c)

Seeking the existence of a product solution, one assumes that the solution is representable as

$$\Theta(x,t) = X(x)Y(t)$$

which will produce the two ordinary differential equations

$$\frac{1}{\alpha Y}\frac{dY}{dt} = \frac{d^2X}{dx^2}\frac{1}{X} = -\lambda^2 \qquad -②$$

where λ is the separation parameter and the negative sign was chosen to assure a negative exponential solution in time

$$Y = a_1 e^{-\alpha\lambda^2 t} \quad X = a_2\cos\lambda x + a_3\sin\lambda x$$

Thus, the general solution to equation ① is

$$\theta(x,t) = (C_1\cos\lambda x + C_2\sin\lambda x)e^{-\alpha\lambda^2 t} \qquad -③$$

The constants C_1, C_2, and λ are to be determined from the imposed boundary conditions and initial condition

From B.C. (a), $0 = C_1 e^{-\alpha\lambda^2 t}$ and $C_1=0$ for $t>0$

From B.C. (b), $0 = (C_2 \sin\lambda L)e^{-\alpha\lambda^2 t}$ where C_2 cannot be zero because it may result in a trivial solution when $\begin{cases}\Theta = 0 \\ t > 0 \\ x = 0\end{cases}$

Thus $\sin\lambda L = 0$ where $\lambda = \frac{n\pi}{L}$, where n=1,2,3⋯

(note : n=0 is excluded because it would give the trivial solution, $\Theta=0$)

The sum of the corresponding solutions for each λ is the solution in equation ③

$$\theta(x,t) = \sum_{n=1}^{\infty} C_n e^{-[n\pi/L]^2 \alpha t} \sin\frac{n\pi x}{L} \qquad \text{-④}$$

Apply the initial condition at t=0 as given by (c) to obtain

$$\theta_i = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \qquad \text{-⑤}$$

What are the values of C_n that satisfy equation ⑤ ?

Determination of the C_n's requires an application of the theory of Fourier sine series of an odd function.

$$C_n = \frac{2}{L}\int_0^L \theta_i \sin\frac{n\pi x}{L} dx = \frac{4}{n\pi}\theta_i \qquad n = 1,3,5\cdots$$
$$= 0 \qquad n = 2,4,6\cdots$$

NOTE : *Fourier* sine *series of an odd function having length of L*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

The theory of Fourier series states that an arbitrary function f(x) may be expressed as an infinite series of sine functions of the form

In our case $f(x) = \theta_i = T_i - T_1$

$$\therefore C_n = \frac{2}{L}\int_0^L \theta_i \sin\frac{n\pi x}{L}dx = \frac{4}{n\pi}\theta_i = \frac{4}{n\pi}(T_i - T_1) \quad \text{for} \quad n = 1,3,5\cdots$$

Finally, substitute C_n into equation ④

$$\theta(x,t) = \frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\theta_i e^{-[n\pi/L]^2\alpha t}\sin\frac{n\pi x}{L}$$

$$\frac{\theta}{\theta_i} = \frac{T - T_1}{T_i - T_1} = \frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}e^{-[n\pi/L]^2\alpha t}\sin\frac{n\pi x}{L} \quad \text{for } n = 1, 3, 5 \ldots$$

4-2 LUMPED HEAT CAPACITY SYSTEM

We shall continue our discussion of transient heat conduction by assuming that the temperature of system is only a function of time and is uniform throughout the system at any instant. The above assumption (or simplification) is justified when the external thermal resistance (or surface-convection resistance) between the surface of the system and the surrounding medium is so large compared to the internal thermal resistance (or internal-conduction resistance) of the system that it controls the heat transfer process. As a result, the major temperature gradient would occur through the fluid layer at the surface.

A measure of the relative importance of the thermal resistance within a solid body is the ratio of the internal resistance to the external resistance, called the Biot number, B_i. It is defined as

$$B_i = \frac{R_i}{R_e} = \frac{hL}{k}$$

where h : the average heat transfer coefficient ($W/m^2 \cdot °C$)

$L = V/A_s$: characteristic dimension of solid body

k : thermal conductivity of solid body

Consider the sudden cooling of a small metal casting or a billet in a quenching bath after its removal from a hot furnace so that the temperature change by a step will be experienced

Change in an internal energy of the billet during dt (Black's heat capacity equation)	=	Net heat flow from the billet to the quenching bath during dt (Newton's law of cooling)

$$q = -c\rho V\frac{dT}{dt} = hA_s(T - T_\infty) \qquad \text{where} \quad T > T_\infty$$

The convection heat loss from the body is experienced as a decrease in the internal energy of the body.

Separate the variables and integrate with initial condition ($T = T_o$ at $t = 0$)

$$\int_{T_o}^{T}\frac{d(T-T_\infty)}{T-T_\infty}dT = \int_0^t -\frac{hA_s}{c\rho V}dt$$

$$\ln\left(\frac{T-T_\infty}{T_o-T_\infty}\right) = -\left[\frac{hA_s}{c\rho V}\right]t$$

$$\therefore \quad \frac{T-T_\infty}{T_o-T_\infty} = e^{-\left[\frac{hA_s}{c\rho V}\right]t}$$

where, the exponent $\left[\frac{hA_s t}{c\rho V}\right]$ must be dimensionless

$$\frac{hA_s t}{c\rho V} = \left(\frac{hL}{k}\right)\left(\frac{kt}{c\rho L^2}\right) = \left(\frac{hL}{k}\right)\left(\frac{\alpha t}{L^2}\right) = (Bi)(Fo)$$

Fo : **Fourier number** is the ratio of the heat transfer rate by conduction to the energy storage rate in the system. It is an important parameter in transient conduction problem.

The quantity $\left[\frac{c\rho V}{hA_s}\right]$ is called the time constant of the system since it has the dimension of time. Its value is indicative of the rate of response of a single-capacity system to a sudden change in the environmental temperature. Observe that when $t = \left[\frac{c\rho V}{hA_s}\right]$, the temperature difference $T-T_\infty$ is equal to 36.8% of the initial temperature difference T_o-T_∞

Applicability of Lumped Heat Capacity Analysis

We have already noted that the lumped heat capacity analysis assumes a uniform temperature distribution throughout the solid body and that the assumption is equivalent to saying that the surface-convection resistance is large compared with the internal-conduction resistance. Such an analysis may be expected to yield reasonable estimates within about 5% when the following condition is met:

$$hL/k < 0.1$$

The reader should recognize that there are many practical cases where the lumped heat capacity method may yield good results. Examples in Table 4-1 illustrate the relative validity of such cases. We may point out that uncertainties in the heat transfer coefficient of ± 25% are quite common.

Table 4-1 | Examples of lumped-capacity systems.

Physical situation	k, W/m · °C	Approximate value of h, W/m^2 · °C	$\frac{h(V/A)}{k}$
1. 3.0-cm steel cube cooling in room air	40	7.0	8.75×10^{-4}
2. 5.0-cm glass cylinder cooled by a 50-m/s airstream	0.8	180	2.81
3. Same as situation 2 but a copper cylinder	380	180	0.006
4. 3.0-cm hot copper cube submerged in water such that boiling occurs	380	10,000	0.132

Steel Ball Cooling in Air — EXAMPLE 4-1

A steel ball [$c = 0.46$ kJ/kg · °C, $k = 35$ W/m · °C] 5.0 cm in diameter and initially at a uniform temperature of 450°C is suddenly placed in a controlled environment in which the temperature is maintained at 100°C. The convection heat-transfer coefficient is 10 W/m^2 · °C. Calculate the time required for the ball to attain a temperature of 150°C.

Solution

We anticipate that the lumped-capacity method will apply because of the low value of h and high value of k. We can check by using Equation (4-6):

$$\frac{h(V/A)}{k} = \frac{(10)[(4/3)\pi(0.025)^3]}{4\pi(0.025)^2(35)} = 0.0023 < 0.1$$

so we may use Equation (4-5). We have

$$T = 150°\text{C} \qquad \rho = 7800\ \text{kg/m}^3 \qquad [486\ \text{lb}_m/\text{ft}^3]$$
$$T_\infty = 100°\text{C} \qquad h = 10\ \text{W/m}^2 \cdot °\text{C} \qquad [1.76\text{Btu/h} \cdot \text{ft}^2 \cdot °\text{F}]$$
$$T_0 = 450°\text{C} \qquad c = 460\ \text{J/kg} \cdot °\text{C} \qquad [0.11\ \text{Btu/lb}_m \cdot °\text{F}]$$

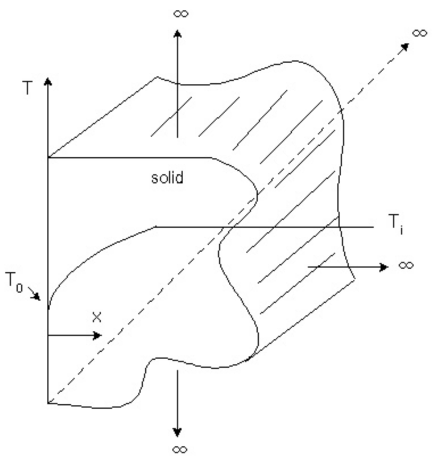
$$\frac{hA}{\rho c V} = \frac{(10)4\pi(0.025)^2}{(7800)(460)(4\pi/3)(0.025)^3} = 3.344 \times 10^{-4}\ \text{s}^{-1}$$

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-[hA/\rho c V]\tau}$$

$$\frac{150 - 100}{450 - 100} = e^{-3.344\times 10^{-4}\tau}$$

$$\tau = 5819\ \text{s} = 1.62\ \text{h}$$

4-3 TRANSIENT HEAT FLOW IN A SEMI-INFINITE SOLID (very thick wall and infinitely long solid)



Consider the semi-infinite solid shown in the above figure maintained at some initial temperature T_i. The surface temperature is suddenly lowered and maintained at temperature T_0 and we seek an expression for the temperature distribution in the solid as a function of time.

Heat diffusion equation :

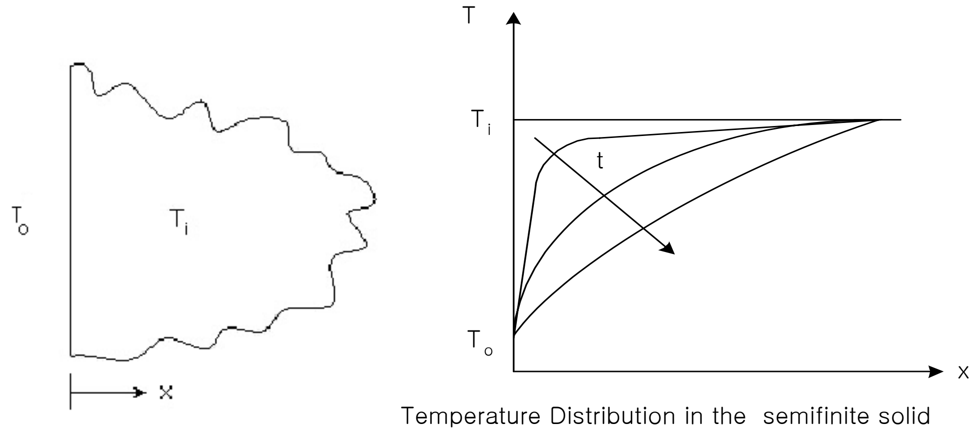
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{B.C.} \quad T(0,t)=T_0, \quad T(\infty,t) = T_i$$

$$\text{I.C.} \quad T(x,0)=T_i$$

For the initial condition we shall specify that the temperature inside the solid is uniform at T_i, that is, $T(x,0)=T_i$. For one of the two required boundary conditions, we postulate that far from the surface the interior temperature will not be affected by the temperature wave,
i.e., $T(\infty,t) = T_i$

The closed-form solutions have been obtained for four types of changes in surface conditions, instantaneously applied at t=0

(Case 1) A Sudden Change of Temperature at Surface : the surface temperature is suddenly lowered and maintained at T_o at x=0



Temperature Distribution in the semifinite solid

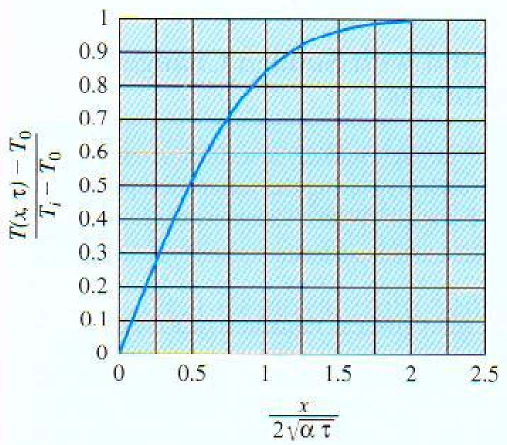
We have $\{B.C.'s : T(0,t) = T_o,\ T(\infty,t) = T_i\}$ and $\{I.C. : T(x,0) = T_i\}$
This is the problem which may be solved by the Laplace-Transform technique and the solution is given as; $\dfrac{T(x,t) - T_o}{T_i - T_o} = erf\left(\dfrac{x}{2\sqrt{\alpha t}}\right)$

where, the Gaussian error function $erf\left(\frac{x}{2\sqrt{\alpha t}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\alpha t}}} e^{-\eta^2} d\eta$

where, η is a dummy variable and the integral is a function of its upper limit.

(Note) : $\frac{d}{dx}\int_0^{\frac{x}{2\sqrt{\alpha t}}} e^{-\eta^2} d\eta = e^{-\frac{x^2}{4\alpha t}} \frac{d}{dx}\left(\frac{x}{2\sqrt{\alpha t}}\right)$

Figure 4-4 | Response of semi-infinite solid to sudden change in surface temperature



The rate of heat flow at the surface (at x=0) becomes

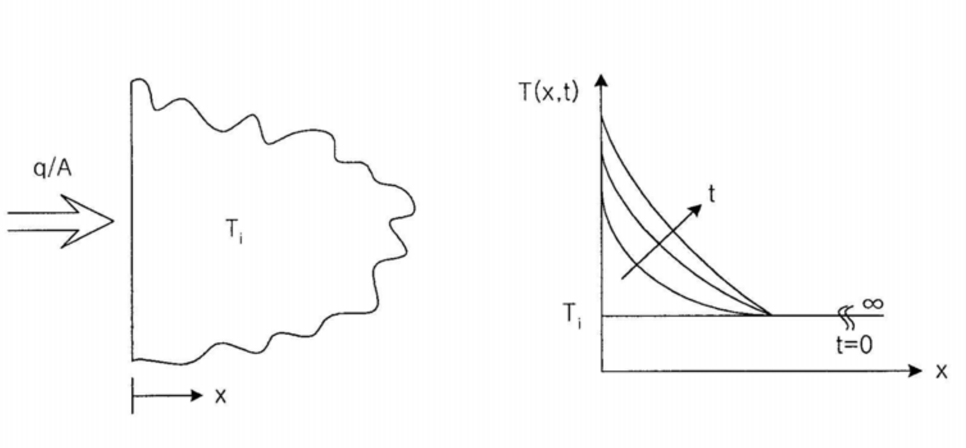
$$q_o = -kA\frac{dT}{dx}\bigg]_{x=0} = \frac{kA(T_o - T_i)}{\sqrt{\pi\alpha t}} \quad \text{and}$$

$$\frac{dT}{dx} = (T_i - T_o)\frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{4\alpha t}}\frac{d}{dx}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \frac{T_i - T_o}{\sqrt{\pi\alpha t}}e^{-\frac{x^2}{4\alpha t}}$$

The total heat transfer at the surface over the period of time

$$Q_o = \int_0^t q_o dt = 2kA(T_o - T_i)\sqrt{\frac{t}{\pi\alpha}} \quad (Joule)$$

(Case 2) Constant Heat Flux at Surface : a sudden application of a specified heat flux $(q/A)_{x=0} = q_o/A$ (the surface is suddenly exposed to the thermal radiation)



We have $\left\{ B.C.'s\ ;\ T(\infty,t) = T_i \quad \frac{q_o}{A} = -k\frac{dT}{dx}\Big]_{x=o} \right\}$ and $\{I.C.\ ;\ T(x,0) = T_i\}$

$$T - T_i = \frac{2q_o\sqrt{\alpha t/\pi}}{kA}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q_o x}{kA}\left(1 - erf\frac{x}{2\sqrt{\alpha t}}\right)$$

Therefore, at x=0 (at the surface) $\quad T_o = \frac{2q_o\sqrt{\alpha t/\pi}}{kA} + T_i$

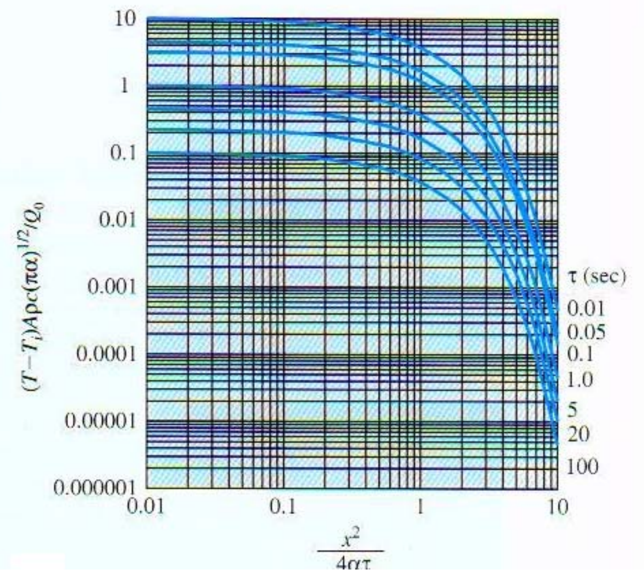
And heat flux at a given position x becomes $\left(\frac{q}{A}\right)_x = -k\frac{dT}{dx}\Big]_x$

(Case 3) Energy Pulse at Surface : a short, instantaneous pulse of energy having a magnitude of $Q_o/A\,(J/m^2)$ is applied at the surface.

The resulting temperature response is given by

$$T - T_i = \left[\frac{Q_o}{A\rho c(\pi\alpha t)^{1/2}}\right]\exp\left(-\frac{x^2}{4\alpha t}\right)$$

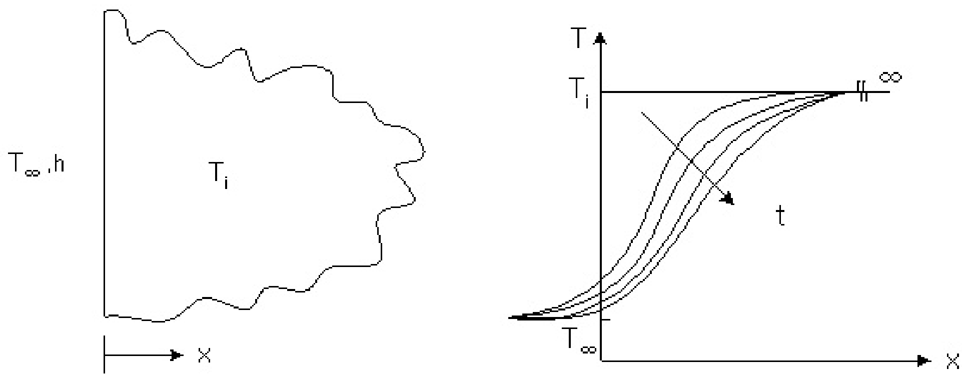
Figure 4-4 | Response of semi-infinite solid to instantaneous surface pulse of Q_0/A J/m^2



In contrast to the constant-heat-flux case where the temperature increases indefinitely for all x and times, the temperature response to the instantaneous surface pulse will die out with time, or $T-T_i \rightarrow 0$ for $all\ x\ as\ t \rightarrow \infty$.

The rapid exponential decay behavior is illustrated in Figure 4-4 above.

(Case 4) Convection Boundary Conditions : a sudden exposure of the surface to a fluid at a different temperature through a uniform heat transfer coefficient h



$$\text{B.C.'S ;}\quad -k\frac{dT}{dx}\bigg]_{x=0} = h(T(0,t) - T_\infty),\quad T(\infty,t) = T_i$$

$$\text{I.C. ;}\quad T(x,0) = T_i$$

$$\frac{T - T_i}{T_\infty - T_i} = 1 - erf\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[1 - erf\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right] \qquad \textbf{[4-15]}$$

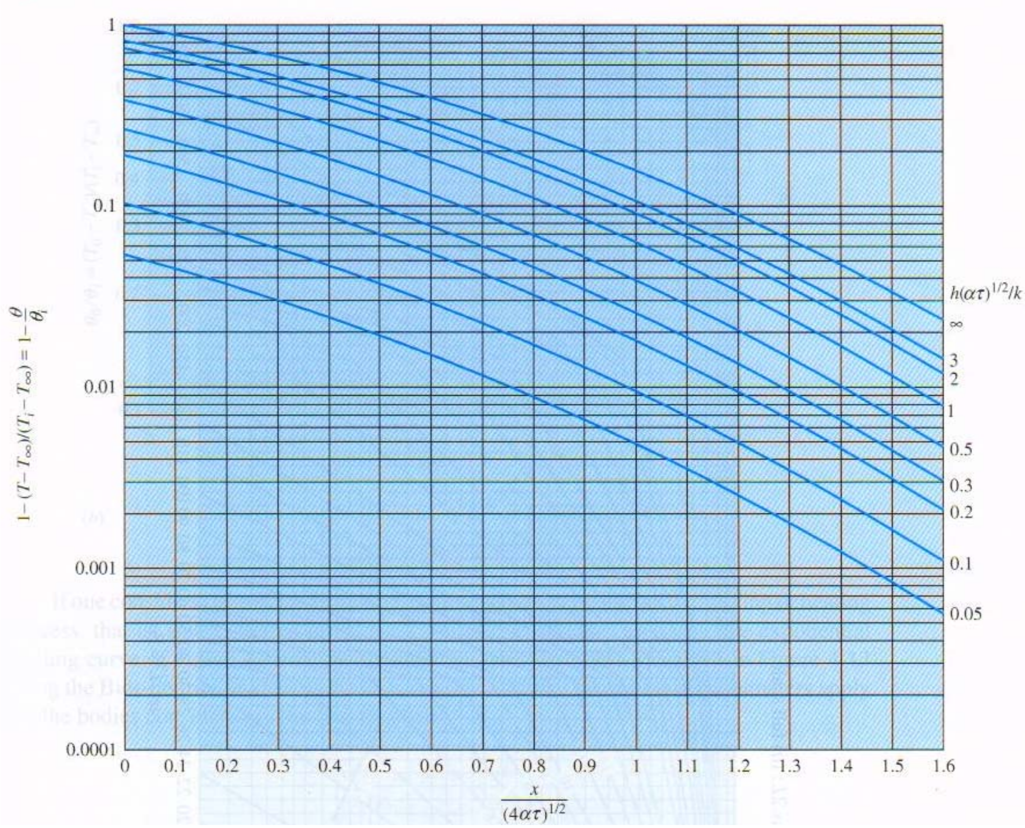
or

$$Since \quad \frac{T - T_i}{T_\infty - T_i} = 1 - \frac{T - T_\infty}{T_i - T_\infty}$$

$$Thus, \frac{T - T_\infty}{T_i - T_\infty} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right) + \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[erfc\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

The solution is presented in graphical in Figure 4-5.

Figure 4-5 | Temperature distribution in the semi-infinite solid with convection boundary condition.



Iterative method(trial-and-error procedure) is used to determine the time required for the surface or interior to reach a certain temperature.

At the surface where x=0, T(0,t) = T_o is plugged into the above solution to obtain

$$T^* = \frac{T_o - T_\infty}{T_i - T_\infty} = e^{\gamma^2} erfc(\gamma) \qquad \text{--- (a)}$$

where, $\gamma = \frac{h\sqrt{\alpha t}}{k} = \frac{h\sqrt{t}}{\sqrt{\rho c k}}$ since $\alpha = \frac{k}{\rho c}$

$$Thus,\ h = \frac{\gamma\sqrt{\rho c k}}{\sqrt{t}}$$

Knowing T_o, T_i, and T_∞, we can calculate T^*. First, make an initial guess for h and calculate γ. Now substitute γ into equation (a) to see if it satisfies the initially guessed value of h. If not, repeat the same procedure until it agrees with a previously guessed value of h.

EXAMPLE 4-2 Semi-Infinite Solid with Sudden Change in Surface Conditions

A large block of steel [$k = 45$ W/m · °C, $\alpha = 1.4 \times 10^{-5}$ m^2/s] is initially at a uniform temperature of 35°C. The surface is exposed to a heat flux (*a*) by suddenly raising the surface temperature to 250°C and (*b*) through a constant surface heat flux of 3.2×10^5 W/m^2. Calculate the temperature at a depth of 2.5 cm after a time of 0.5 min for both these cases.

■ Solution

We can make use of the solutions for the semi-infinite solid given as Equations (4-8) and (4-13*a*). For case *a*,

$$\frac{x}{2\sqrt{\alpha\tau}} = \frac{0.025}{(2)[(1.4 \times 10^{-5})(30)]^{1/2}} = 0.61$$

The error function is determined from Appendix A as

$$\text{erf}\ \frac{x}{2\sqrt{\alpha\tau}} = \text{erf}\ 0.61 = 0.61164$$

We have $T_i = 35°\text{C}$ and $T_0 = 250°\text{C}$, so the temperature at $x = 2.5$ cm is determined from Equation (4-8) as

$$T(x,\tau) = T_0 + (T_i - T_0)\,\text{erf}\ \frac{x}{2\sqrt{\alpha\tau}}$$

$$= 250 + (35 - 250)(0.61164) = 118.5°\text{C}$$

For the constant-heat-flux case *b*, we make use of Equation (4-13*a*). Since q_0/A is given as 3.2×10^5 W/m^2, we can insert the numerical values to give

$$T(x,\tau) = 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} e^{-(0.61)^2}$$

$$- \frac{(0.025)(3.2 \times 10^5)}{45}(1 - 0.61164)$$

$$= 79.3°\text{C} \qquad x = 2.5\ \text{cm},\ \tau = 30\ \text{s}$$

For the constant-heat-flux case the *surface* temperature after 30 s would be evaluated with $x = 0$ in Equation (4-13*a*). Thus,

$$T(x=0) = 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} = 199.4°\text{C}$$

EXAMPLE 4-3 Pulsed Energy at Surface of Semi-Infinite Solid

An instantaneous laser pulse of 10 MJ/m^2 is imposed on a slab of stainless steel having properties of $\rho = 7800\ kg/m^3$, $c = 460\ J/kg \cdot °C$, and $\alpha = 0.44 \times 10^{-5}\ m^2/s$. The slab is initially at a uniform temperature of 40 °C. Estimate the temperature at the surface and at a depth of 2.0 mm after a time of 2 s.

■ Solution

This problem is a direct application of Equation (4-13*b*). We have $Q_0/A = 10^7\ J/m^2$ and at $x = 0$

$$T_0 - T_i = Q_0/A\rho c(\pi\alpha\tau)^{1.2}$$
$$= 10^7/(7800)(460)[\pi(0.44 \times 10^{-5})(2)]^{0.5} = 530°C$$

and

$$T_0 = 40 + 530 = 570°C$$

At $x = 2.0\ mm = 0.002\ m$,

$$T - T_i = (530)\exp[-(0.002)^2/(4)(0.44 \times 10^{-5})(2)] = 473°C$$

and

$$T = 40 + 473 = 513°C$$

Heat Removal from Semi-Infinite Solid

EXAMPLE 4-4

A large slab of aluminum at a uniform temperature of 200°C suddenly has its surface temperature lowered to 70°C. What is the total heat removed from the slab per unit surface area when the temperature at a depth 4.0 cm has dropped to 120°C?

■ Solution

We first find the time required to attain the 120°C temperature and then integrate Equation (4-12) to find the total heat removed during this time interval. For aluminum,

$$\alpha = 8.4 \times 10^{-5}\ \mathrm{m^2/s} \qquad k = 215\ \mathrm{W/m \cdot {}^\circ C}\ [124\ \mathrm{Btu/h \cdot ft \cdot {}^\circ F}]$$

We also have

$$T_i = 200^\circ\mathrm{C} \qquad T_0 = 70^\circ\mathrm{C} \qquad T(x, \tau) = 120^\circ\mathrm{C}$$

Using Equation (4-8) gives

$$\frac{120 - 70}{200 - 70} = \operatorname{erf} \frac{x}{2\sqrt{\alpha\tau}} = 0.3847$$

From Figure 4-4 or Appendix A,

$$\frac{x}{2\sqrt{\alpha\tau}} = 0.3553$$

and

$$\tau = \frac{(0.04)^2}{(4)(0.3553)^2(8.4 \times 10^{-5})} = 37.72\ \mathrm{s}$$

The total heat removed at the surface is obtained by integrating Equation (4-12):

$$\frac{Q_0}{A} = \int_0^\tau \frac{q_0}{A}\,d\tau = \int_0^\tau \frac{k(T_0 - T_i)}{\sqrt{\pi\alpha\tau}}\,d\tau = 2k(T_0 - T_i)\sqrt{\frac{\tau}{\pi\alpha}}$$

$$= (2)(215)(70 - 200)\left[\frac{37.72}{\pi(8.4 \times 10^{-5})}\right]^{1/2} = -21.13 \times 10^6\ \mathrm{J/m^2} \quad [-1861\ \mathrm{Btu/ft^2}]$$

Example 4-5. Sudden Exposure of Semi-Infinite Slab to Convection
The slab of Example 4-4 is suddenly exposed to a convection-surface environment of 70 °C with a heat-transfer coefficient of 525 $W/m^2 \cdot °C$. Calculate the time required for the temperature to reach 120 °C at the depth of 4.0 cm for this circumstance.

■ Solution : we may use Equation (4-15) or Figure 4-5 for this problem, but Figure 4-5 is easier to apply because the time appears in two terms. However, an iterative procedure is required because the time appears in both of the variables $h/\sqrt{\alpha\tau}/k$ and $x/(2\sqrt{\alpha\tau})$.

We seek the value of τ such that

$$\frac{T-T_i}{T_\infty - T_i} = \frac{120-200}{70-200} = 0.615 \qquad [a]$$

We therefore try values of τ and obtain readings of the temperature ratio from Figure 4-5 until agreement with Equation (a) is reached. The iterations are listed below. Values of k and α are obtained from Example 4-4.

τ, s	$\frac{h\sqrt{\alpha\tau}}{k}$	$\frac{x}{2\sqrt{\alpha\tau}}$	$\frac{T-T_i}{T_\infty - T_i}$ from Figure 4-5
1000	0.708	0.069	0.41
3000	1.226	0.040	0.61
4000	1.416	0.035	0.68

Consequently, the time required is approximately 3000 s.

Solutions have been worked out for other geometries. The most important cases are those dealing with (1) plates whose thickness is small in relation to the other dimensions, (2) cylinders where the diameter is small compared to the length, and (3) spheres. Results of analyses for these geometries have been presented in graphical form by Heisler [2], and nomenclature for the three cases is illustrated in Figure 4-6. In all cases the convection environment temperature is designated as T_∞ and the center temperature for $x = 0$ or $r = 0$ is T_0. At time zero, each solid is assumed to have a uniform initial temperature T_i. Temperatures in the solids are given in Figures 4-7 to 4-13 as functions of time and spatial position. In these charts we note the definitions

$$\theta = T(x, \tau) - T_\infty \qquad \text{or} \qquad T(r, \tau) - T_\infty$$

$$\theta_i = T_i - T_\infty$$

$$\theta_0 = T_0 - T_\infty$$

If a centerline temperature is desired, only one chart is required to obtain a value for θ_0 and then T_0. To determine an off-center temperature, two charts are required to calculate the product

$$\frac{\theta}{\theta_i} = \frac{\theta_0}{\theta_i}\frac{\theta}{\theta_0}$$

For example, Figures 4-7 and 4-10 would be employed to calculate an off-center temperature for an infinite plate.

The heat losses for the infinite plate, infinite cylinder, and sphere are given in Figures 4-14 to 4-16, where Q_0 represents the initial internal energy content of the body in reference to the environment temperature

$$Q_0 = \rho c V(T_i - T_\infty) = \rho c V \theta_i \qquad \textbf{[4-16]}$$

In these figures Q is the actual heat lost by the body in time τ.

Figure 4-6 | Nomenclature for one-dimensional solids suddenly subjected to convection environment at T_∞: (*a*) infinite plate of thickness $2L$; (*b*) infinite cylinder of radius r_0; (*c*) sphere of radius r_0.

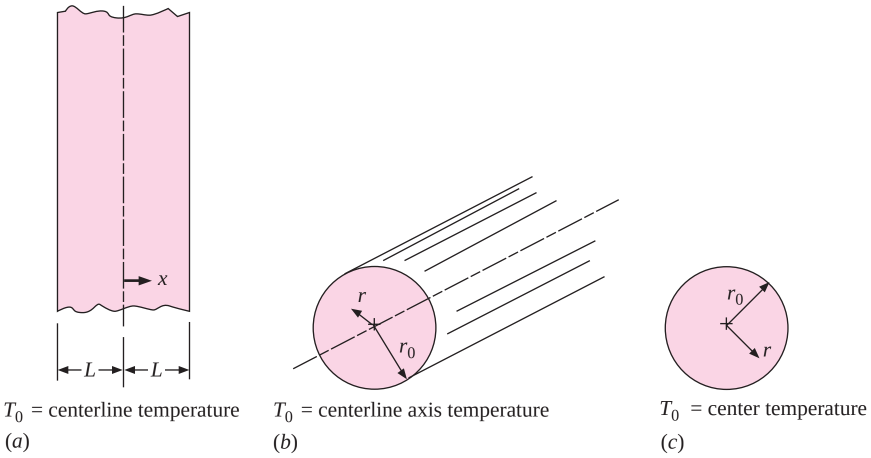
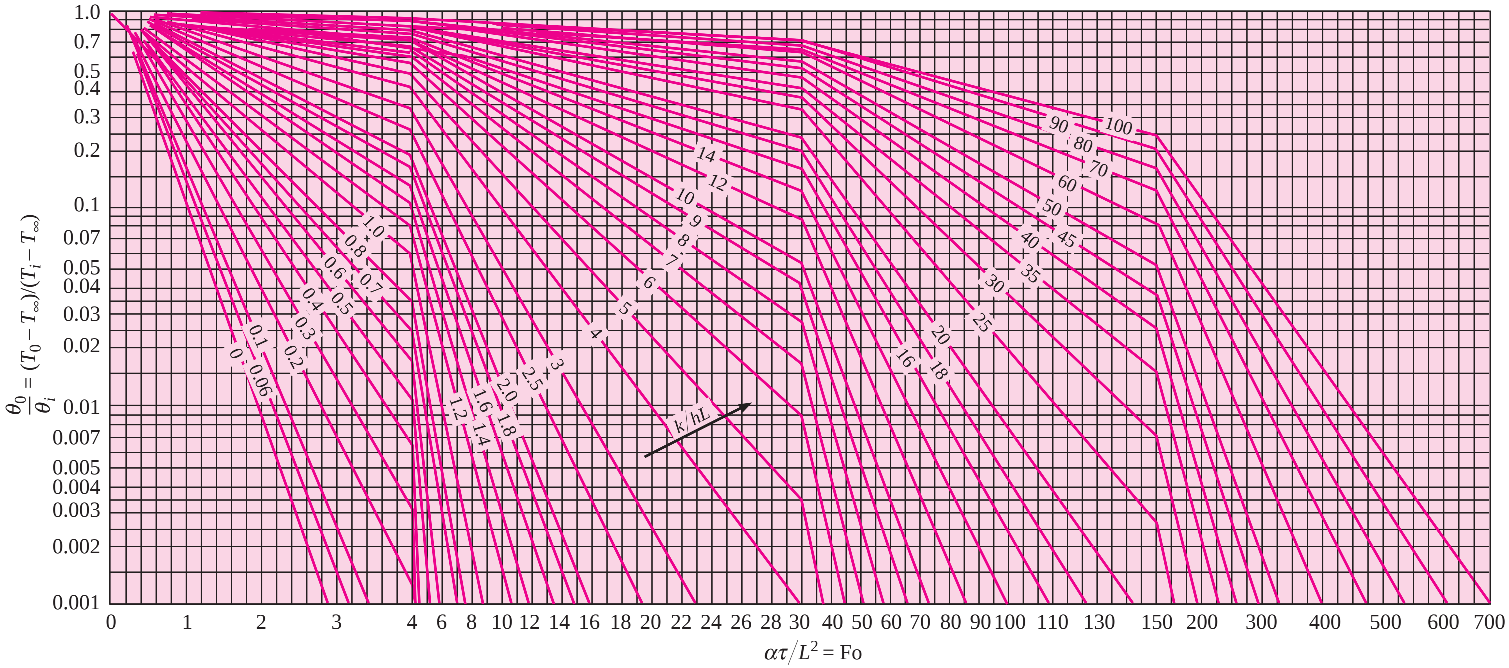
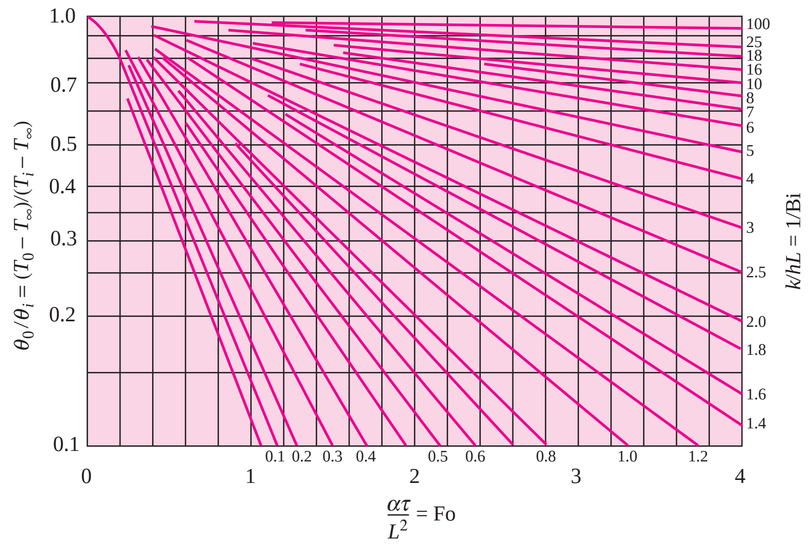


Figure 4-7 | Midplane temperature for an infinite plate of thickness $2L$: (a) full scale.



(a)

Figure 4-7 | (*Continued*). (*b*) expanded scale for 0 < Fo < 4, from Reference 2.



(*b*)

If one considers the solid as behaving as a lumped capacity during the cooling or heating process, that is, small internal resistance compared to surface resistance, the exponential cooling curve of Figure 4-5 may be replotted in expanded form, as shown in Figure 4-13 using the Biot-Fourier product as the abscissa. We note that the following parameters apply for the bodies considered in the Heisler charts.

$$(A/V)_{\text{inf plate}} = 1/L$$
$$(A/V)_{\text{inf cylinder}} = 2/r_0$$
$$(A/V)_{\text{sphere}} = 3/r_0$$

Obviously, there are many other practical heating and cooling problems of interest. The solutions for a large number of cases are presented in graphical form by Schneider [7], and readers interested in such calculations will find this reference to be of great utility.

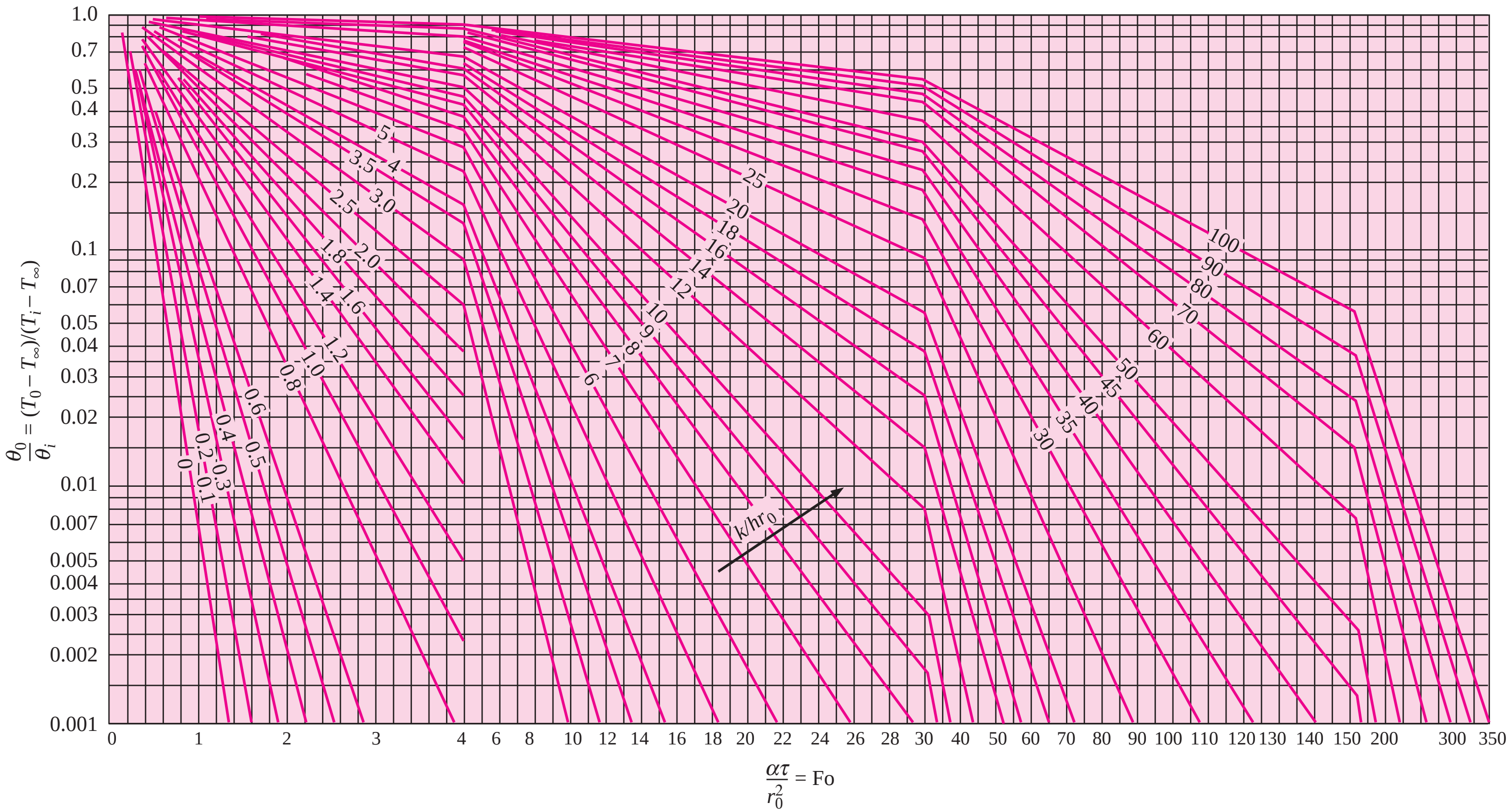
The Biot and Fourier Numbers

A quick inspection of Figures 4-5 to 4-16 indicates that the dimensionless temperature profiles and heat flows may all be expressed in terms of two dimensionless parameters called the Biot and Fourier numbers:

$$\text{Biot number} = \text{Bi} = \frac{hs}{k}$$
$$\text{Fourier number} = \text{Fo} = \frac{\alpha\tau}{s^2} = \frac{k\tau}{\rho c s^2}$$

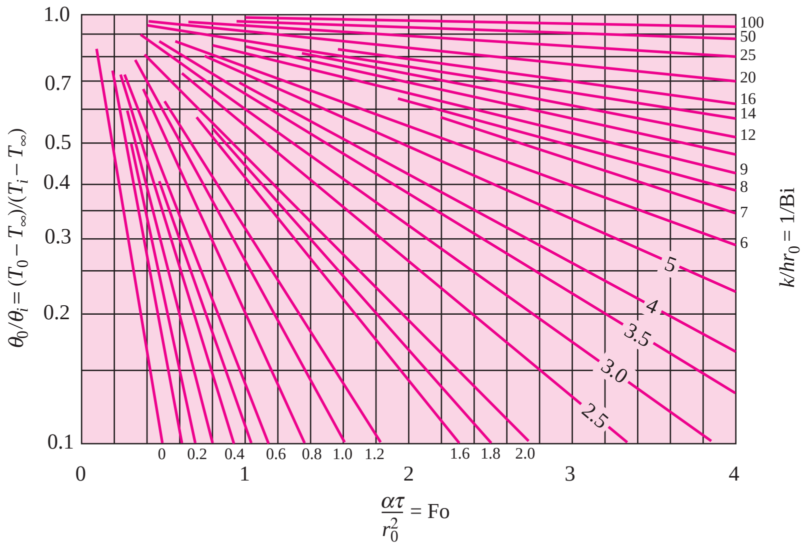
In these parameters s designates a characteristic dimension of the body; for the plate it is the half-thickness, whereas for the cylinder and sphere it is the radius. The Biot number compares the relative magnitudes of surface-convection and internal-conduction resistances

Figure 4-8 | Axis temperature for an infinite cylinder of radius r_0: (a) full scale.



(a)

Figure 4-8 | (*Continued*). (*b*) expanded scale for 0 < Fo < 4, from Reference 2.



(*b*)

to heat transfer. The Fourier modulus compares a characteristic body dimension with an approximate temperature-wave penetration depth for a given time τ.

A very low value of the Biot modulus means that internal-conduction resistance is negligible in comparison with surface-convection resistance. This in turn implies that the temperature will be nearly uniform throughout the solid, and its behavior may be approximated by the lumped-capacity method of analysis. It is interesting to note that the exponent of Equation (4-5) may be expressed in terms of the Biot and Fourier numbers if one takes the ratio V/A as the characteristic dimension s. Then,

$$\frac{hA}{\rho cV}\tau = \frac{h\tau}{\rho cs} = \frac{hs}{k}\frac{k\tau}{\rho cs^2} = \text{Bi Fo}$$

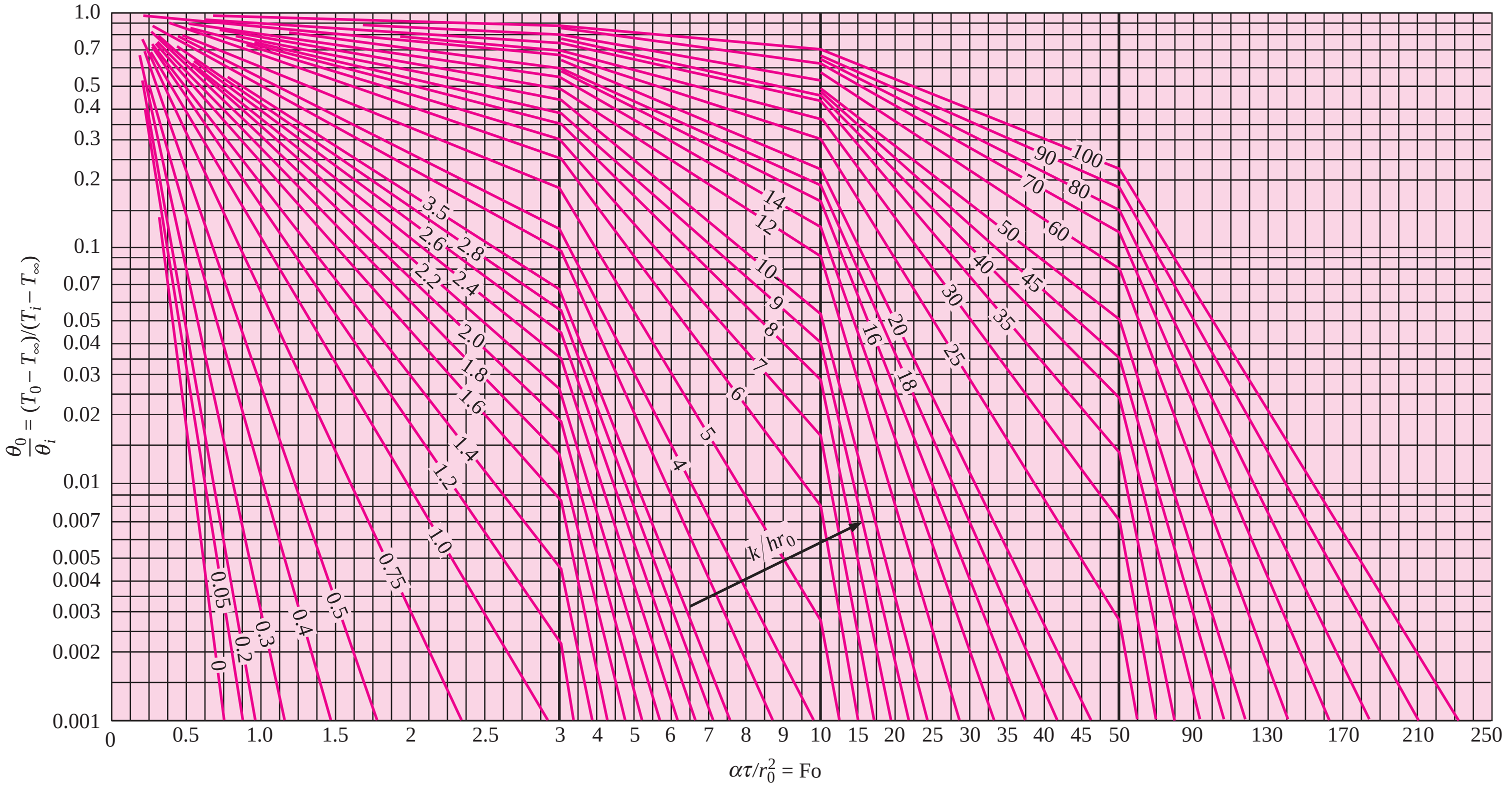
Applicability of the Heisler Charts

The calculations for the Heisler charts were performed by truncating the infinite series solutions for the problems into a few terms. This restricts the applicability of the charts to values of the Fourier number greater than 0.2.

$$\text{Fo} = \frac{\alpha\tau}{s^2} > 0.2$$

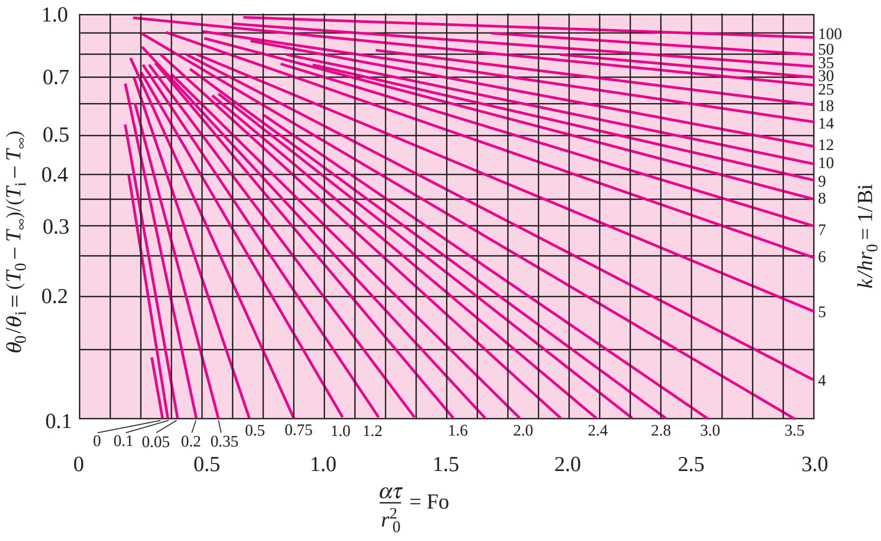
For smaller values of this parameter the reader should consult the solutions and charts given in the references at the end of the chapter. Calculations using the truncated series solutions directly are discussed in Appendix C.

Figure 4-9 | Center temperature for a sphere of radius r_0: (*a*) full scale.



(*a*)

Figure 4-9 | (*Continued*). (*b*) expanded scale for 0 < Fo < 3, from Reference 2.



(*b*)

Figure 4-10 | Temperature as a function of center temperature in an infinite plate of thickness $2L$, from Reference 2.

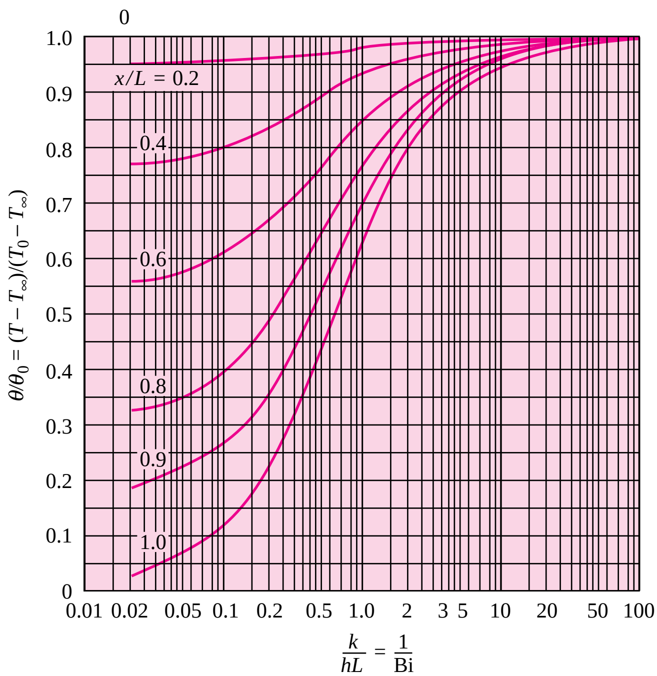


Figure 4-11 | Temperature as a function of axis temperature in an infinite cylinder of radius r_0, from Reference 2.

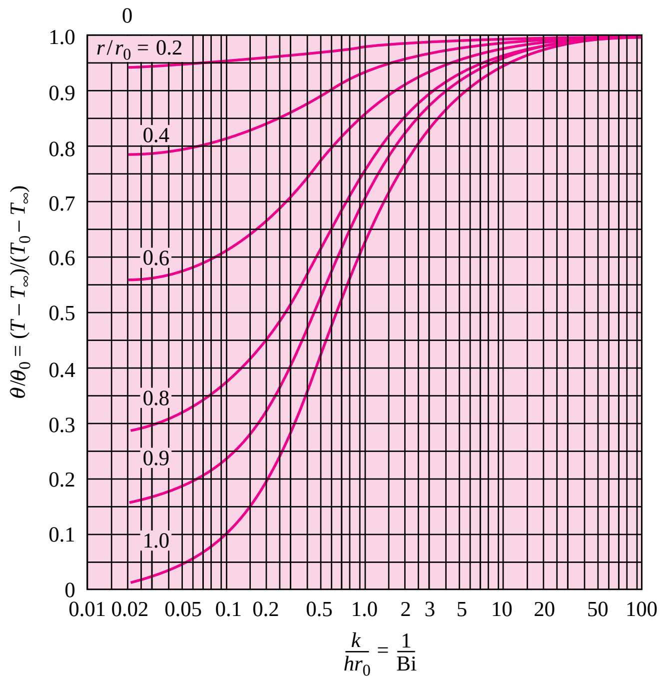


Figure 4-12 | Temperature as a function of center temperature for a sphere of radius r_0, from Reference 2.

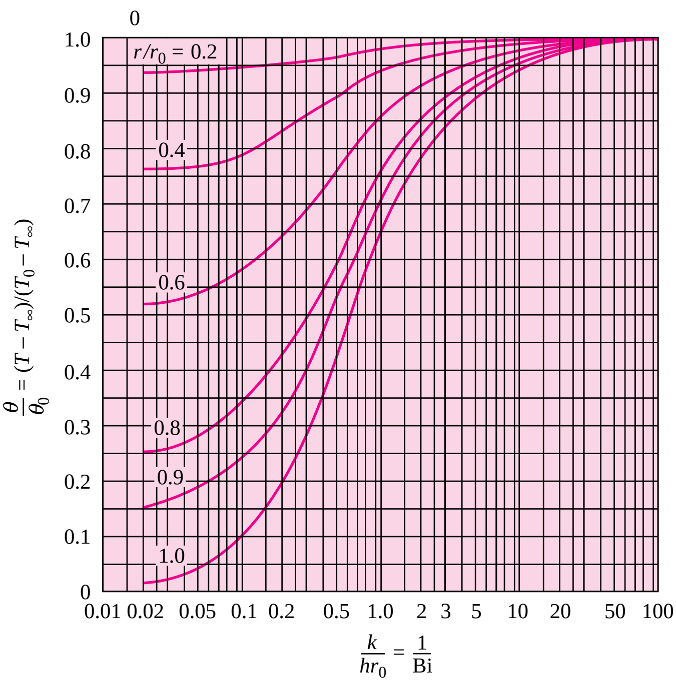
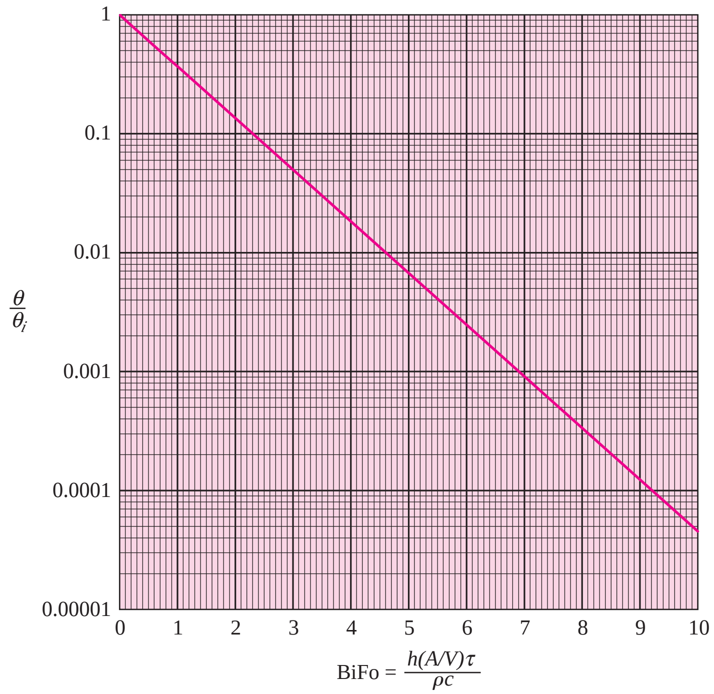
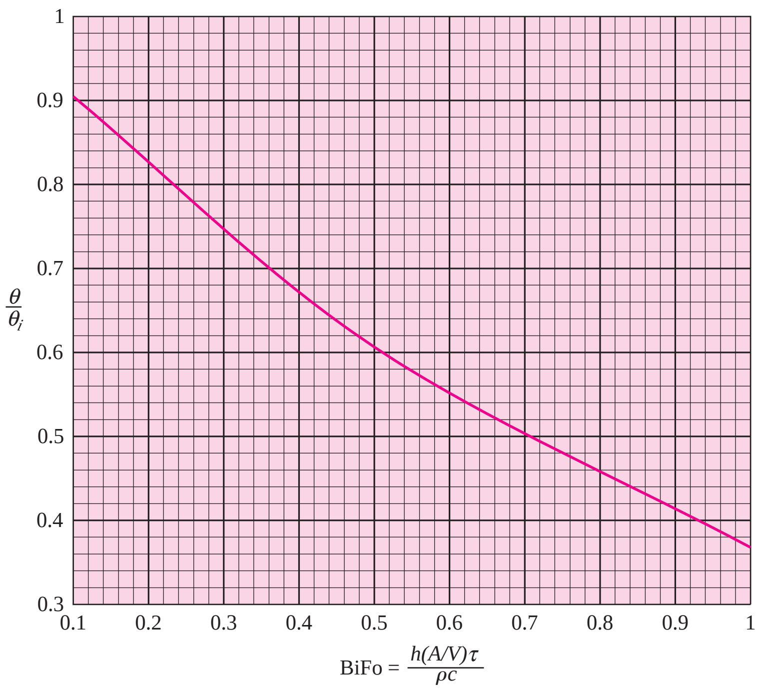


Figure 4-13 | Temperature variation with time for solids that may be treated as lumped capacities: (*a*) $0 < \text{BiFo} < 10$, (*b*) $0.1 < \text{BiFo} < 1.0$, (*c*) $0 < \text{BiFo} < 0.1$. Note: $(A/V)_{\text{inf plate}} = 1/L$, $(A/V)_{\text{inf cyl}} = 2/r_0$, $(A/V)_{\text{sphere}} = 3/r_0$. See Equations (4-5) and (4-6).

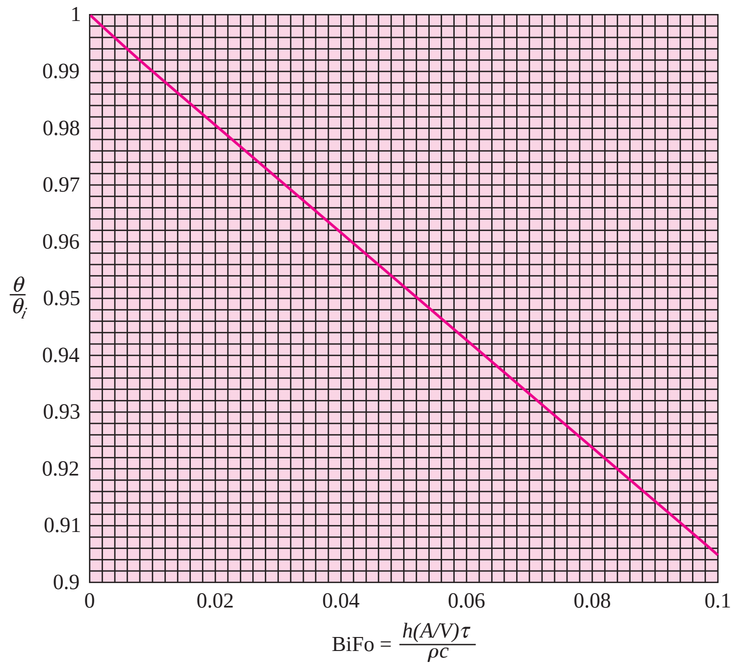


(*a*)



(*b*)

Figure 4-13 | (*Continued*).



(*c*)

Figure 4-14 | Dimensionless heat loss Q/Q_0 of an infinite plane of thickness $2L$ with time, from Reference 6.

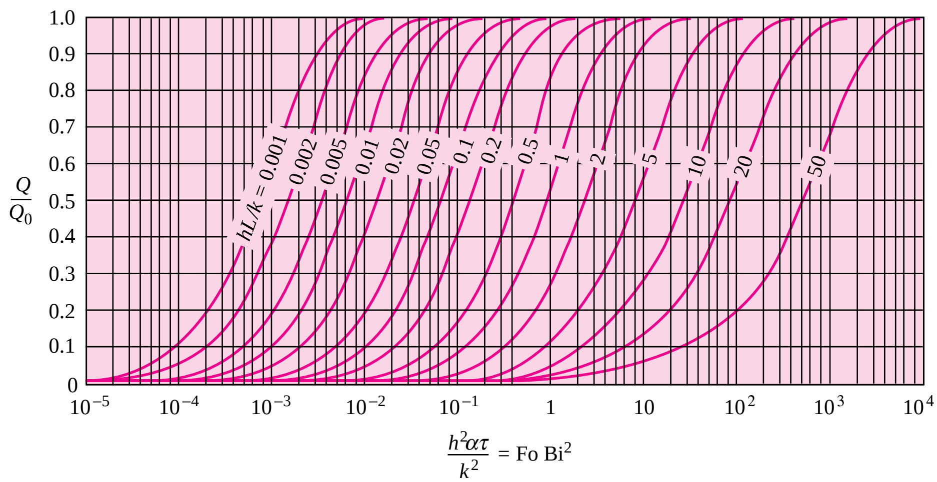


Figure 4-15 | Dimensionlesss heat loss Q/Q_0 of an infinite cylinder of radius r_0 with time, from Reference 6.

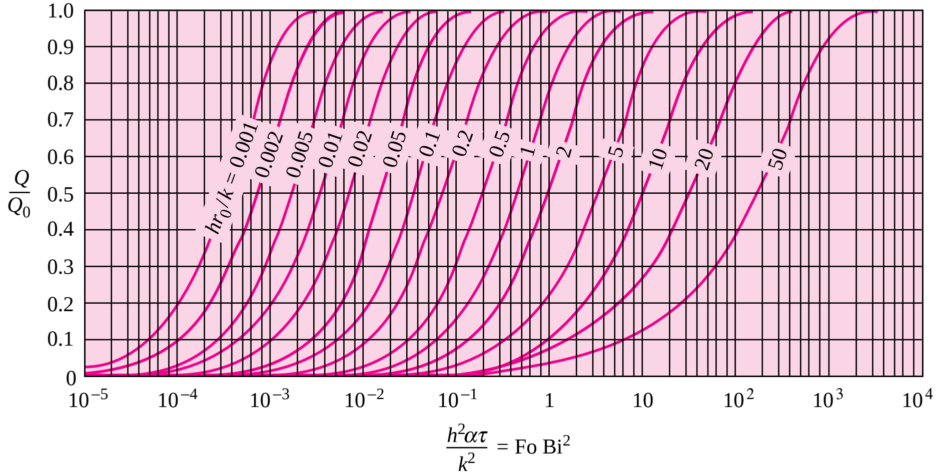
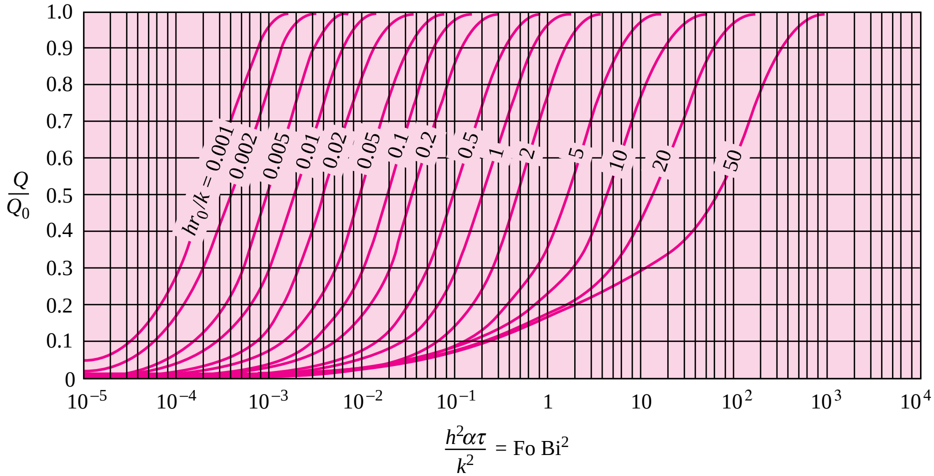


Figure 4-16 | Dimensionless heat loss Q/Q_0 of a sphere of radius r_0 with time, from Reference 6.



EXAMPLE 4-6 Aluminum Plate Suddenly Exposed to Convection

A large plate of aluminum 5.0 cm thick and initially at 200°C is suddenly exposed to the convection environment of Example 4-5. Calculate the temperature at a depth of 1.25 cm from one of the faces 1 min after the plate has been exposed to the environment. How much energy has been removed per unit area from the plate in this time?

■ Solution

The Heisler charts of Figures 4-7 and 4-10 may be used for solution of this problem. We first calculate the center temperature of the plate, using Figure 4-7, and then use Figure 4-10 to calculate the temperature at the specified x position. From the conditions of the problem we have

$$\theta_i = T_i - T_\infty = 200 - 70 = 130 \qquad \alpha = 8.4 \times 10^{-5}\ \text{m}^2/\text{s} \quad [3.26\ \text{ft}^2/\text{h}]$$

$$2L = 5.0\ \text{cm} \qquad L = 2.5\ \text{cm} \qquad \tau = 1\ \text{min} = 60\ \text{s}$$

$$k = 215\ \text{W/m}\cdot{}^\circ\text{C} \quad [124\ \ \text{Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F}]$$

$$h = 525\ \text{W/m}^2\cdot{}^\circ\text{C} \quad [92.5\ \text{Btu/h}\cdot\text{ft}^2\cdot{}^\circ\text{F}]$$

$$x = 2.5 - 1.25 = 1.25\ \text{cm}$$

Then

$$\frac{\alpha\tau}{L^2} = \frac{(8.4\times10^{-5})(60)}{(0.025)^2} = 8.064 \qquad \frac{k}{hL} = \frac{215}{(525)(0.025)} = 16.38$$

$$\frac{x}{L} = \frac{1.25}{2.5} = 0.5$$

From Figure 4-7

$$\frac{\theta_0}{\theta_i} = 0.61$$

$$\theta_0 = T_0 - T_\infty = (0.61)(130) = 79.3$$

From Figure 4-10 at $x/L = 0.5$,

$$\frac{\theta}{\theta_0} = 0.98$$

and

$$\theta = T - T_\infty = (0.98)(79.3) = 77.7$$

$$T = 77.7 + 70 = 147.7^\circ\text{C}$$

We compute the energy lost by the slab by using Figure 4-14. For this calculation we require the following properties of aluminum:

$$\rho = 2700 \text{ kg/m}^3 \qquad c = 0.9 \text{ kJ/kg} \cdot {}^\circ\text{C}$$

For Figure 4-14 we need

$$\frac{h^2 \alpha \tau}{k^2} = \frac{(525)^2 (8.4 \times 10^{-5})(60)}{(215)^2} = 0.03 \qquad \frac{hL}{k} = \frac{(525)(0.025)}{215} = 0.061$$

From Figure 4-14

$$\frac{Q}{Q_0} = 0.41$$

For unit area

$$\begin{aligned} \frac{Q_0}{A} &= \frac{\rho c V \theta_i}{A} = \rho c (2L) \theta_i \\ &= (2700)(900)(0.05)(130) \\ &= 15.8 \times 10^6 \text{ J/m}^2 \end{aligned}$$

so that the heat removed per unit surface area is

$$\frac{Q}{A} = (15.8 \times 10^6)(0.41) = 6.48 \times 10^6 \text{ J/m}^2 \quad [571 \text{ Btu/ft}^2]$$

Long Cylinder Suddenly Exposed to Convection

EXAMPLE 4-7

A long aluminum cylinder 5.0 cm in diameter and initially at 200°C is suddenly exposed to a convection environment at 70°C and $h = 525$ W/m$^2 \cdot$ °C. Calculate the temperature at a radius of 1.25 cm and the heat lost per unit length 1 min after the cylinder is exposed to the environment.

■ Solution

This problem is like Example 4-6 except that Figures 4-8 and 4-11 are employed for the solution. We have

$$\theta_i = T_i - T_\infty = 200 - 70 = 130 \qquad \alpha = 8.4 \times 10^{-5} \text{ m}^2/\text{s}$$
$$r_0 = 2.5 \text{ cm} \qquad \tau = 1 \text{ min} = 60 \text{ s}$$
$$k = 215 \text{ W/m} \cdot {}^\circ\text{C} \qquad h = 525 \text{ W/m}^2 \cdot {}^\circ\text{C} \qquad r = 1.25 \text{ cm}$$
$$\rho = 2700 \text{ kg/m}^3 \qquad c = 0.9 \text{ kJ/kg} \cdot {}^\circ\text{C}$$

We compute

$$\frac{\alpha \tau}{r_0^2} = \frac{(8.4 \times 10^{-5})(60)}{(0.025)^2} = 8.064 \qquad \frac{k}{h r_0} = \frac{215}{(525)(0.025)} = 16.38$$
$$\frac{r}{r_0} = \frac{1.25}{2.5} = 0.5$$

From Figure 4-8

$$\frac{\theta_0}{\theta_i} = 0.38$$

and from Figures 4-11 at $r/r_0 = 0.5$

$$\frac{\theta}{\theta_0} = 0.98$$

so that

$$\frac{\theta}{\theta_i} = \frac{\theta_0}{\theta_i}\frac{\theta}{\theta_0} = (0.38)(0.98) = 0.372$$

and

$$\theta = T - T_\infty = (0.372)(130) = 48.4$$

$$T = 70 + 48.4 = 118.4^\circ\text{C}$$

To compute the heat lost, we determine

$$\frac{h^2\alpha\tau}{k^2} = \frac{(525)^2(8.4\times10^{-5})(60)}{(215)^2} = 0.03 \qquad \frac{hr_0}{k} = \frac{(525)(0.025)}{215} = 0.061$$

Then from Figure 4-15

$$\frac{Q}{Q_0} = 0.65$$

For unit length

$$\frac{Q_0}{L} = \frac{\rho c V\theta_i}{L} = \rho c\pi r_0^2\theta_i = (2700)(900)\pi(0.025)^2(130) = 6.203\times10^5\ \text{J/m}$$

and the actual heat lost per unit length is

$$\frac{Q}{L} = (6.203\times10^5)(0.65) = 4.032\times10^5\ \text{J/m} \quad [116.5\ \text{Btu/ft}]$$