3장 정상상태의 전도 – 2차원

3-1. 서론

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가정: 2차원, 정상, No 발열원

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau} \quad \Rightarrow \frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$ **(Laplace 방정식)** – ①

∘ 방정식 ①을 푸는 3가지 방법
1) 수학적 해석   2)도식적 해석   3)수치적 해석

3-2. 수학적 해석

식①은 2차원 선형(Linear) 미분방정식이므로 해의 중첩(혹은 **선형)**원리 – **superposition(or linearity) principle of solutions**– 를 적용할 수 있고, 변수분리법(**Separation of variables method**)을 사용된다.

변수분리법의 핵심은 식①의 해가 다음의 두 개의 해의 곱이라고 가정함

$$T = X \cdot Y$$ –②

여기에서 $X = X(x)$이고 $Y = Y(y)$이다

이제 식②를 ①에 대입하여 다음을 얻는다.

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \; ; \quad \lambda = \text{분리변수}$$ –③

Now, we can obtain two following ordinary differential equations

$$\begin{cases} \dfrac{d^2X}{dx^2} + \lambda^2 X = 0 \\ \dfrac{d^2Y}{dy^2} - \lambda^2 Y = 0 \end{cases} \qquad -④$$

The value of λ^2 must be determined from the boundary conditions.

The solutions to eqn ④ are

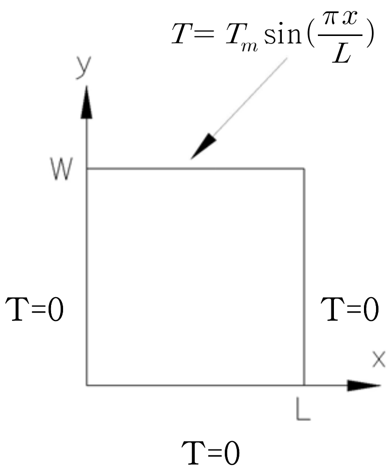
$X = A\cos\lambda x + B\sin\lambda x$

$Y = Ce^{-\lambda y} + De^{\lambda y}$ where $\begin{cases} e^{-\lambda y} = \cosh\lambda y - \sinh\lambda y \\ e^{\lambda y} = \cosh\lambda y + \sinh\lambda y \end{cases}$

Finally, 일반해 **(the general solution)** is

$$T = (A\cos\lambda x + B\sin\lambda x)(Ce^{-\lambda y} + De^{\lambda y}) \qquad -⑤$$

(Case 1) Rectangular adiabatic plate with sinusoidal temperature distribution on one edge.



T = 0 at y = 0

T = 0 at x = 0

T = 0 at x = L

$T = T_m \sin(\frac{\pi x}{L})$ at y = W

Substituting these conditions into eqn ⑤ to obtain the solution for T
We obtain from the first condition

$$(A\cos\lambda x + B\sin\lambda x)(C+D) = 0 \qquad -⑥$$

From the second condition, we obtain

$$A(Ce^{-\lambda y} + De^{\lambda y}) = 0 \qquad -⑦$$

From the third condition, we obtain

$$(A\cos\lambda L + B\sin\lambda L)(Ce^{-\lambda y} + De^{\lambda y}) = 0 \qquad -⑧$$

Eqn. ⑥ can be satisfied only if D= −C, and eqn. ⑦ only if A=0
Substituting these results into eqn. ⑧ gives

$$(B\sin\lambda L)(Ce^{-\lambda y} + De^{\lambda y}) = -2BC\sin\lambda L(\frac{e^{\lambda y} - e^{-\lambda y}}{2})$$

$$= -2BC\sin\lambda L\sinh\lambda y = 0 \qquad -⑨$$

To satisfy this condition, $\sin\lambda L$ must be zero, and $\sin\lambda L = 0$ results in $\lambda = \frac{n\pi}{L}$, where n = 1,2,3... (It should be noted that the value n=0 is excluded because it would give a trivial solution, T=0)

Thus,

$$T = (A\cos\lambda x + B\sin\lambda x)(Ce^{-\lambda y} + De^{\lambda y}) = B\sin\lambda x(Ce^{-\lambda y} - Ce^{\lambda y})$$

$$= -2BC\sin\lambda x\ \sinh\lambda y \quad \text{where } \lambda = \frac{n\pi}{L} \ (n=1,2,3.......)$$

There exists therefore a different solution for each integer ***n*** and each solution has a separate integration constant **c*n*** (the constants are combined into ***Cn***), summing these solutions, we get

$$T = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} sinh\frac{n\pi y}{L} \qquad -⑩$$

Now, apply the last boundary condition to obtain

$$T_m \sin(\frac{\pi x}{L}) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} sinh\frac{n\pi W}{L}$$

which requires that C_n = 0 for n>1. The final solution is therefore

$$T_m \sin(\frac{\pi x}{L}) = C_1 \sin\frac{\pi x}{L} sinh\frac{\pi W}{L} \qquad \therefore C_1 = \frac{T_m}{\sinh\frac{\pi W}{L}}$$

Finally, the solution becomes

$$T(x,y) = T_m \frac{\sinh(\frac{\pi y}{L})}{\sinh(\frac{\pi W}{L})} sin(\frac{\pi x}{L})$$

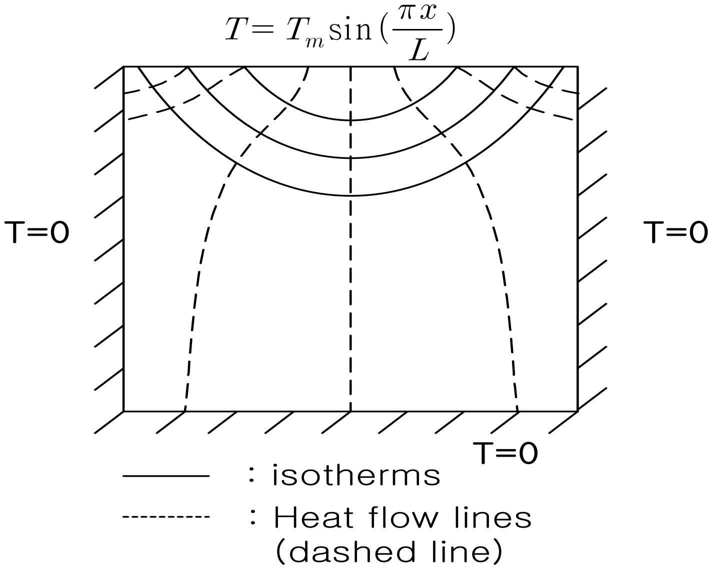
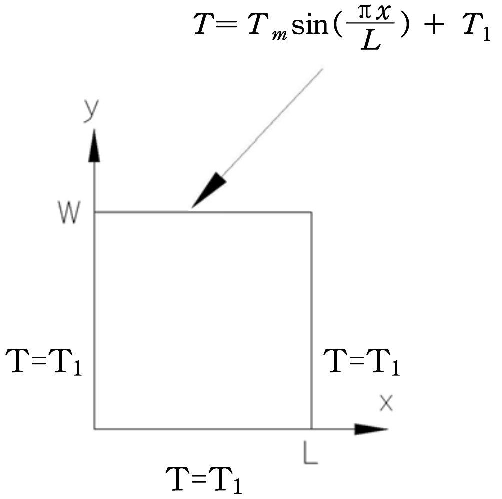


Fig. Isotherms and heat flow lines for the adiabatic plate with sinusoidal temperature distribution on one edge

Note : lines indicating the direction of heat flow are perpendicular to the isotherms

(Case 1-1)



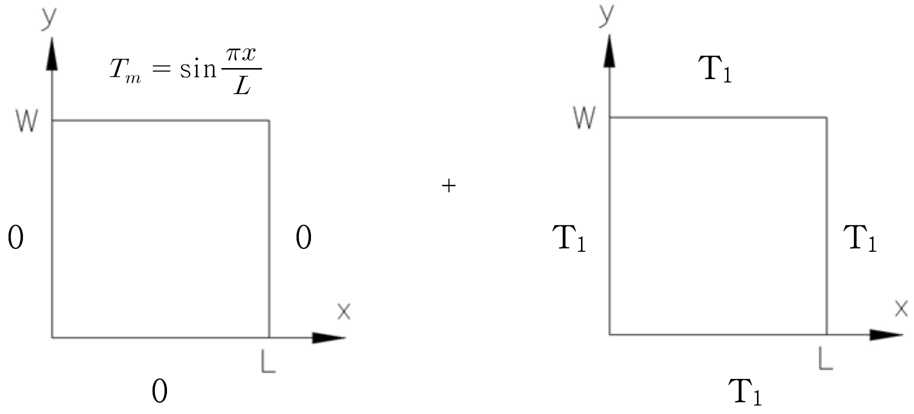
$T = T_1$ at $x = 0$

$T = T_1$ at $y = 0$

$T = T_1$ at $x = L$

$T = T_m \sin(\frac{\pi x}{L}) + T_1$ at $y=W$

Use the superposition principle below to solve this problem.

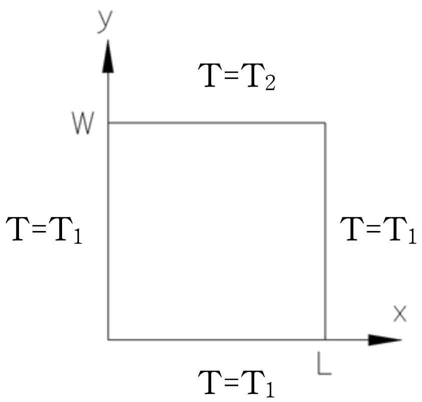


Thus, the solution to this problem is as follows

$$T(x,y) = T_m \frac{\sinh(\frac{\pi y}{L})}{\sinh(\frac{\pi W}{L})} sin(\frac{\pi x}{L}) + T_1$$

(Case 2) Rectangular plate with one edge at a uniform temperature, all other edges at constant temperature

$T = T_1$ at $x = 0$,
$T = T_1$ at $y = 0$,
$T = T_1$ at $x = L$,
$T = T_2$ at $y = W$



Let $\Theta = T - T_1$

$\Theta = 0$ at $x = 0$,
$\Theta = 0$ at $y = 0$,
$\Theta = 0$ at $x = L$,
$\Theta = \Theta_2$ at $y = W$

The general solution to this problem is given as

$$\theta = (A\cos\lambda x + B\sin\lambda x)(Ce^{-\lambda y} + De^{\lambda y})$$

Apply the first 3 boundary conditions and obtain

$$\theta = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} sinh\frac{n\pi y}{L} \qquad -⑪$$

And apply the fourth boundary condition into eqn ⑪ to obtain

$$\theta_2 = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\,\sinh\frac{n\pi W}{L}$$

Need to determine C_n. This is a Fourier sine series of odd function and the value of C_n may be determined by expanding Θ_2 in a Fourier series over the internal 0 < x < L

$$\frac{\theta_2}{\sinh\frac{n\pi W}{L}} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}$$

where, $C_n = \frac{2}{L}\int_0^L \frac{\theta_2}{\sinh\frac{n\pi W}{L}} \sin\frac{n\pi x}{L} dx = \frac{2}{L}\frac{\theta_2}{\sinh\frac{n\pi W}{L}}\int_0^L \sin\frac{n\pi x}{L} dX$

$$= \frac{2}{L}\frac{\theta_2}{\sinh\frac{n\pi W}{L}}\left[-\frac{L}{n\pi}cos\frac{n\pi x}{L}\right]_0^L = \frac{4}{n\pi}\frac{\theta_2}{\sinh\frac{n\pi W}{L}} \quad \text{when n=1, 3, 5...}$$

$$= 0 \quad \text{when n=2, 4, 6..}$$

Now, substitute the above into eqn ⑪ to obtain

$$\theta = \sum_{n=1,3,5...}^{\infty} \frac{4}{n\pi}\frac{\theta_2}{\sinh(\frac{n\pi W}{L})} sin(\frac{n\pi}{L})\sinh(\frac{n\pi y}{L})$$

$$\frac{\theta}{\theta_2} = \frac{T - T_1}{T_2 - T_1} = \sum_{n=1,3,5...}^{\infty} \frac{4}{n\pi} sin(\frac{n\pi x}{L}) \frac{\sinh(\frac{n\pi y}{L})}{\sinh(\frac{n\pi W}{L})} \qquad -⑫$$

In other form as in the textbook (eqn 3-20 on page 81)

$$\frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi W}{L}\right)} \qquad \text{-(13)}$$

NOTE 1: Fourier sine series of an odd function f(x) having a length L

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

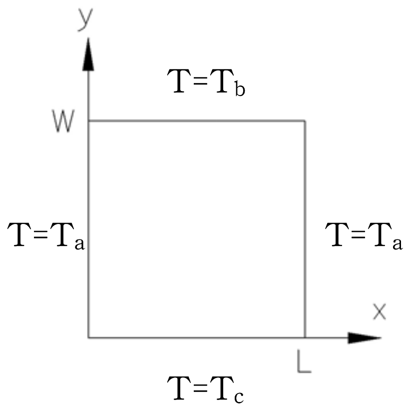
$$Then,\ b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

NOTE 2: Fourier cosine series of an even function f(x) with a length L

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

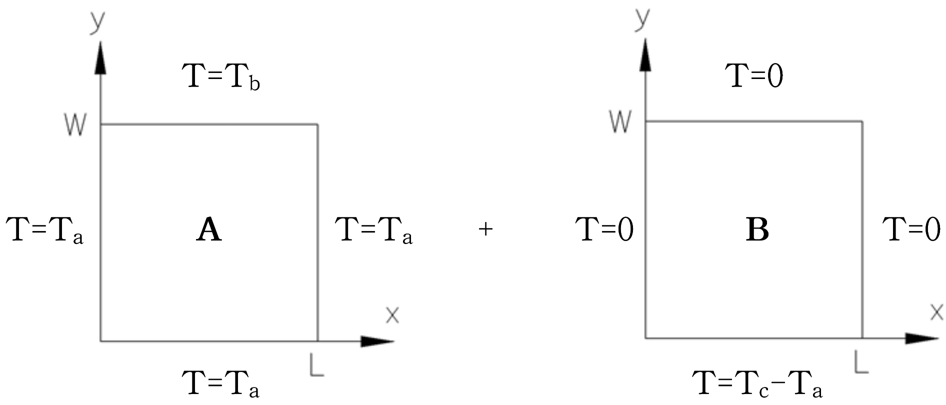
$$Then,\ a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx$$

(Case 3) The rectangular plate with more than one edge at a specified temperature



The figure illustrates a case in which the edge at y = 0 and y = W are held at two different temperatures T_b. and T_c., respectively, while the other edges are maintained at T_a. Since the governing equation for the temperature distribution is linear, the additive principle of superposition may be applied.

Thus, the temperature distribution may be obtained by **adding** the solution obtained by replacing $T_2 = T_b$ and $T_1 = T_a$ in eqn (13) **to** the solution obtained from replacing $T_2 = T_c - T_a$, $T_1 = 0$, and $y = W - y$ in eqn (13)



Solution for A is given as

$$\frac{T-T_a}{T_b-T_a}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}sin(\frac{n\pi x}{L})\frac{\sinh(\frac{n\pi y}{L})}{\sinh(\frac{n\pi W}{L})}$$

Solution for B is given as

$$\frac{T}{T_c-T_a}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}sin(\frac{n\pi x}{L})\frac{\sinh(\frac{n\pi(W-y)}{L})}{\sinh(\frac{n\pi W}{L})}$$

Thus, solution for A + B becomes

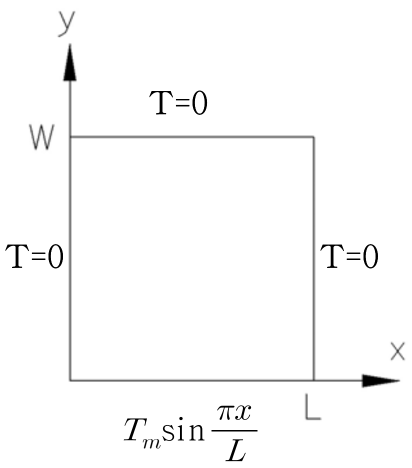
$$T = \frac{2(T_b - T_a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} sin(\frac{n\pi x}{L}) \frac{\sinh(\frac{n\pi y}{L})}{\sinh(\frac{n\pi W}{L})}$$

$$+ \frac{2(T_c - T_a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} sin(\frac{n\pi x}{L}) \frac{\sinh(\frac{n\pi (W-y)}{L})}{\sinh(\frac{n\pi W}{L})}$$

$$= \frac{2}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} \frac{\sinh(\frac{n\pi x}{L})}{\sinh(\frac{n\pi W}{L})} \right\} \left\{ (T_b - T_a)\sinh(\frac{n\pi y}{L}) + (T_c - T_a)\sinh(\frac{n\pi (W-y)}{L}) \right\}$$

$$+ T_a$$

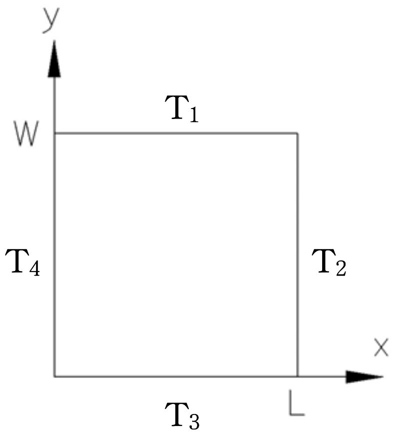
(Case1-A) A sinusoidal temperature distribution is applied at y=0. It is similar to Case1 where the temperature distribution is applied at y=W.



Solution: y needs to be replaced by W-y in the solution to **Case 1**

Thus, $T(x,y) = T_m \dfrac{\sinh \dfrac{\pi(W-y)}{L}}{\sinh \dfrac{\pi W}{L}} sin \dfrac{\pi x}{L}$

(Case 4) Four sides of the rectangular are at different temperature



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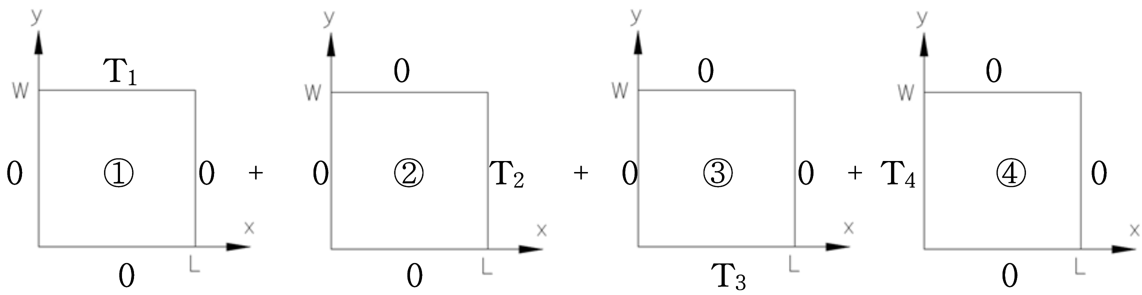

$\begin{cases} T_2 \ is \ replaced \ by \ T_1 \\ T_1 = 0 \ at \ solution \\ \text{for } Case2 - Eq(13) \end{cases}$ $\begin{cases} x \ exchnge \ with \ y \\ w \ exchnge \ with \ L \\ T_1 \ is \ replaced by \ T_2 \\ in \ the \ solution \ 1 \end{cases}$ $\begin{cases} y \ is \ replaced by \ W-y \\ T_1 \ is \ replaced by \ T_3 \\ in \ the \ solution \ 1 \end{cases}$ $\begin{cases} x \ is \ replaced by \ L-x \\ T_2 \ is \ replaced by \ T_4 \\ in \ the \ solution \ 2 \end{cases}$

3-3 도식적 해석

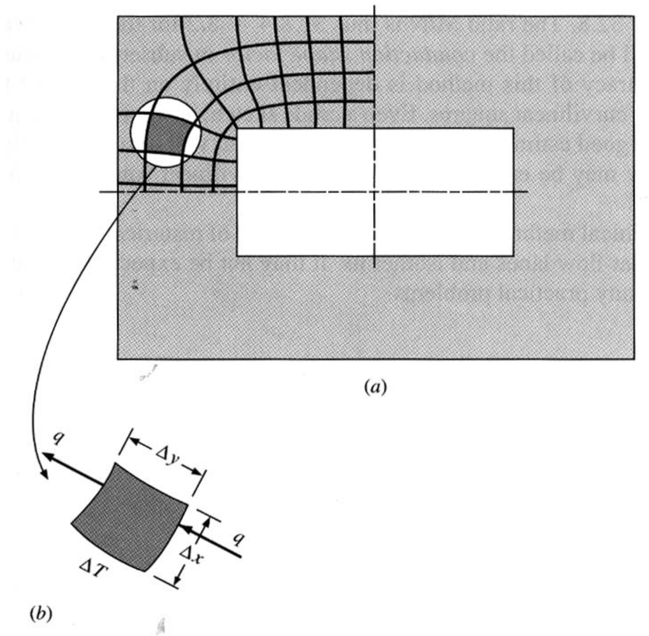

Figure 3-3a shows an element used for curvilinear square($\triangle x \approx \triangle y$) analysis of two-dimensional heat flow.

The heat flow across the curvilinear section (Fig. 3-3b) with the depth L of material is given by Fourier's law

$$q = -k\Delta x L \frac{\Delta T}{\Delta y}$$

The heat flow is the same through each section within this heat-flow lane and the total heat flow is the sum of the heat flows through all the lanes.

If $\Delta x \approx \Delta y$, $q = -kL\Delta T$

Since the heat flow is constant, the ΔT across each element must be the same within the same heat-flow lane. And the ΔT across one element is given by

$$\Delta T = \frac{\Delta T_{all}}{N} = \frac{T_1 - T_2}{N}$$

where N is the number of temperature increments between the inner surface at T_1 and the outer surface at T_2

Now, the heat transfer rate for one heat flow lane becomes

$$q = kL\frac{\Delta T_{all}}{N}$$

And for M heat flow lanes, the total heat transfer rate becomes

$$q_{total} = \frac{M}{N}kL\Delta T_{all} \quad \text{where, } S = \frac{ML}{N} \text{(conduction shape factor)}$$

Thus, $q_{total} = kS\Delta T_{all} = \dfrac{\Delta T_{all}}{1/kS}$ and a thermal resistance due to a 2-dimensional heat conduction is $R_{(2D\ cond)} = 1/kS$

3-4 The Conduction Shape Factor (열전도 형상계수, $S = \frac{ML}{N}$)

In order to calculate the heat transfer rate in the two-dimensional system having a depth of L, we need to construct curvilinear square plots as shown in Fig. 3-3 and count the number of temperature increments and heat-flow lanes. Care must be taken to construct the plot so that $\Delta x \approx \Delta y$ and the lines are perpendicular.

For a convenience of analysis we consider only 1/4 section in the corner of the entire body in Fig. 3-3. The number of temperature increments between the inner and outer surfaces is about N = 4, while the number of heat-flow lanes for the corner section may be estimated as M = 8.2. The total number of heat-flow lanes is four times this value, or 4 x 8.2 = 32.8. The ratio M/N is thus 32.8/4 = 8.2 for the whole body. This ratio will be called the conduction shape factor per unit depth.

The accuracy of this method is dependent entirely on the skill of the person sketching the curvilinear squares. Even a crude sketch, however, can frequently help to give fairly good estimates of the temperatures that will occur in a body. The values of S have been worked out for several geometries and are summarized in **Table 3-1 on pages 84-86**.

3-4-1 Procedure for Constructing the Curviliner Square Plot

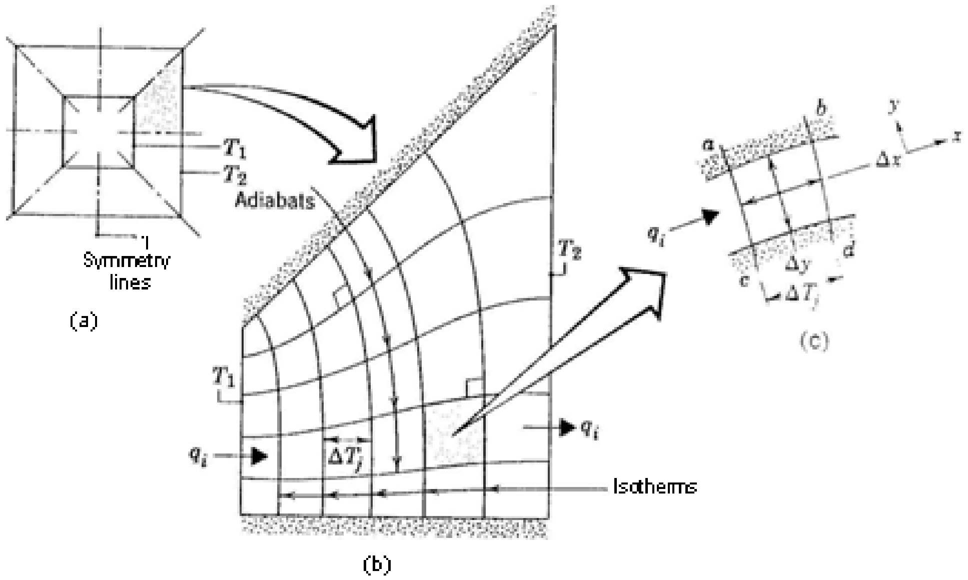

Figure 3-4-1. Two-dimensional conduction in a square channel of length L (a) symmetry planes, (b) curvilinear square plot, (c) typical curvilinear square
* There are 5 heat flow lanes (M=5) and 6 isotherms (N=6)

Fig. 3-4-1 shows a square, two-dimensional channel whose inner and outer surfaces are maintained at T_1 and T_2, respectively. A cross section of the channel is shown in Fig. 3-4-1(a). A procedure for constructing the curvilinear-square plots is enumerated as follows.

1. Identify all relevant lines of symmetry. Such lines are determined by thermal, as well as geometrical conditions. For the square channel of Fig. 3-4-1(a), such lines include the designated vertical, horizontal, and diagonal lines. For this system it is therefore possible to consider only one-eighth of the configuration, as shown in Fig. 3-4-1(b).

2. Lines of symmetry are adiabatic in the sense that there can be no heat transfer in a direction perpendicular to the lines. They are therefore heat

flow lines and should be treated as such. Since there is no heat flow in a direction perpendicular to a heat flow line, such a line can be termed an adiabat.

3. After all known lines of constant temperature associated with the system boundaries have been identified, an attempt should be made to sketch lines of constant temperature within the system. Note that isotherms should always be perpendicular to adiabats.

4. The heat flow lines should then be drawn with an eye toward creating a network of curvilinear squares. This is done by having the heat flow lines and isotherms intersect at right angles and by requiring $\triangle x \approx \triangle y$. It is often impossible to satisfy this second requirement exactly, and it is more realistic to strive for equivalence between the sums of the opposite sides of each square, as shown in Fig. 3-4-1(c). Assigning the x coordinate to the direction of heat flow and the y coordinate to the direction normal to this flow, the requirement may be expressed as

$$\triangle x \equiv \frac{ab+cd}{2} \approx \triangle y \equiv \frac{ac+bd}{2}$$

It is difficult to create a satisfactory network of curvilinear squares in the first attempt, and numerous iterations must often be made. This trial-and-error process involves adjusting the isotherms and adiabats until satisfactory curvilinear squares are obtained for most of the network. Once the curvilinear-square plot has been obtained, it may be used to infer the temperature distribution in the medium. From a simple analysis, the heat transfer rate may then be obtained.