

## Section 6.5 Least-Squares Problem

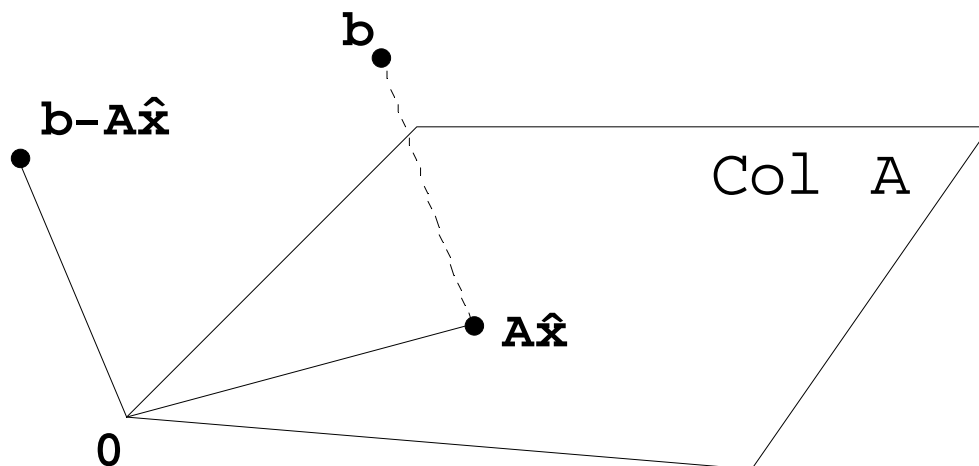
**Problem:** What do we do when  $A\mathbf{x} = \mathbf{b}$  has no solution  $\mathbf{x}$ ?

**Answer:** Find  $\hat{\mathbf{x}}$  such that  $A\hat{\mathbf{x}}$  is as “close” as possible to  $\mathbf{b}$ . (*Least Squares Problem*)

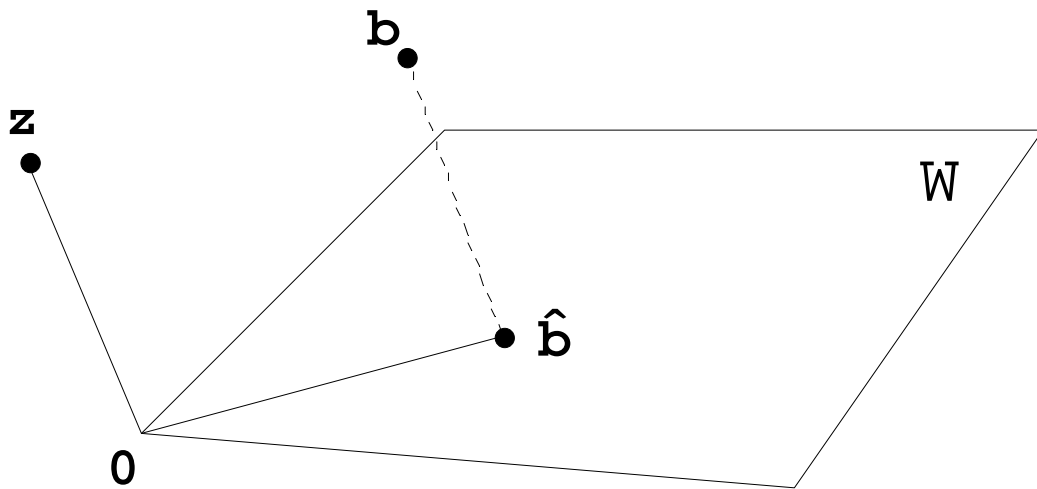
If  $A$  is  $m \times n$  and  $\mathbf{b}$  is in  $\mathbf{R}^m$ , a **least-squares solution** of  $A\mathbf{x} = \mathbf{b}$  is an  $\hat{\mathbf{x}}$  in  $\mathbf{R}^n$  such that

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$$

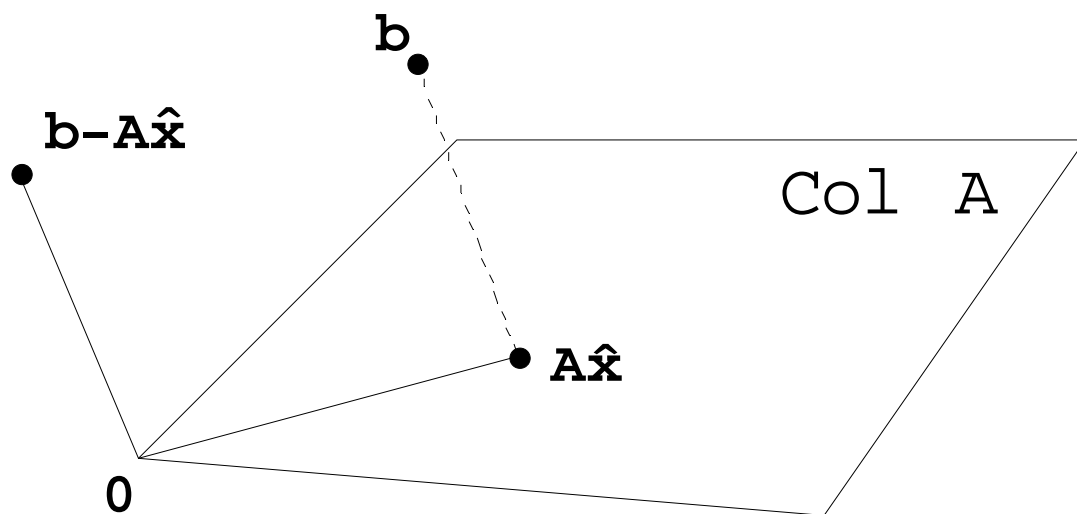
for all  $\mathbf{x}$  in  $\mathbf{R}^n$ .



Let  $W = \text{Col } A$  where  $A$  is  $m \times n$  and  $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ .  
 Suppose  $\mathbf{b}$  is in  $\mathbf{R}^m$  and  $\hat{\mathbf{b}} = \text{proj}_W \mathbf{b}$ .



$\hat{\mathbf{b}}$  is the point in  $W = \text{Col } A$  closest to  $\mathbf{b}$



Since  $\hat{\mathbf{b}}$  is in  $\text{Col } A$ , then  $\hat{\mathbf{x}}$  is a vector in  $\mathbf{R}^n$  such that  $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$ .

By the Orthogonal Projection Theorem,  $\mathbf{z}$  is in  $W^\perp$  where  $\mathbf{z} = \mathbf{b} - A\hat{\mathbf{x}}$ .

Then  $\mathbf{b} - A\hat{\mathbf{x}}$  is orthogonal to every column of  $A$ . Meaning that

$$\mathbf{a}_1^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \quad \mathbf{a}_2^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \quad \dots \quad \mathbf{a}_n^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0$$

$$\begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} (\mathbf{b} - A\hat{\mathbf{x}}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$$

$$A^T\mathbf{b} - A^TA\hat{\mathbf{x}} = \mathbf{0}$$

$$\boxed{A^TA\hat{\mathbf{x}} = A^T\mathbf{b}}$$

(normal equations for  $\hat{\mathbf{x}}$ )

## THEOREM 13

The set of least squares solutions of  $A\mathbf{x} = \mathbf{b}$  is the set of all solutions of the normal equations  $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ .

**EXAMPLE:** Find a least squares solution to the inconsistent system  $A\mathbf{x} = \mathbf{b}$  where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

*Solution:* Solve  $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$  after first finding  $A^T A$  and  $A^T \mathbf{b}$ .

$$A^T A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

So solve the following:

$$\underbrace{\begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underbrace{\begin{bmatrix} 8 \\ 8 \end{bmatrix}}$$

$$\begin{bmatrix} 8 & 2 & 8 \\ 4 & 3 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$$

When  $A^T A$  is invertible,

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$$

$$(A^T A)^{-1} A^T A \hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$$

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$$

So in the last example,

$$(A^T A)^{-1} = \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{16} & -\frac{1}{8} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

and

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} \frac{3}{16} & -\frac{1}{8} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$$

## THEOREM 14

The matrix  $A^T A$  is invertible if and only if the columns of  $A$  are linearly independent. In this case, the equation  $A\mathbf{x} = \mathbf{b}$  has only one least-squares solution  $\hat{\mathbf{x}}$ , and it is given by

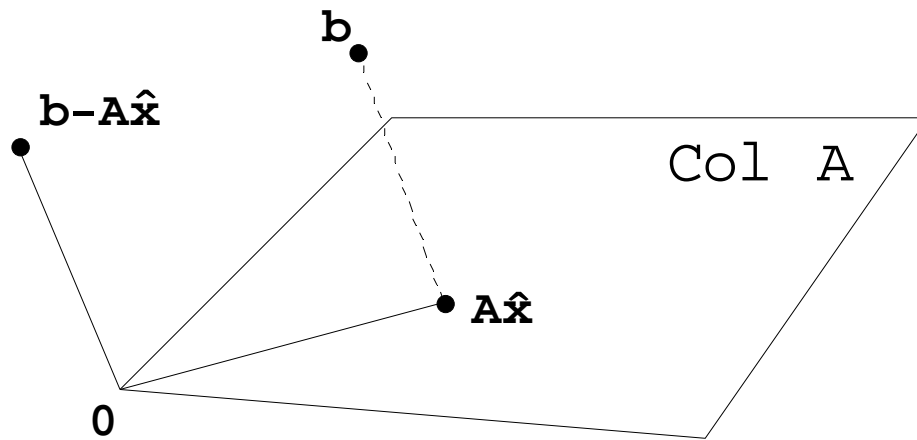
$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}.$$

$$\text{least-squares error} = \|\mathbf{b} - A\hat{\mathbf{x}}\|$$

From the last example,

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } A\hat{\mathbf{x}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\text{least-squares error} = \|\mathbf{b} - A\hat{\mathbf{x}}\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \right\| = 2$$



For another way to compute  $\hat{\mathbf{x}}$ , see Theorem 15 (page 414) and Example 5, page 415.