

Chapter 4

Calculating the Derivative

JMerrill, 2009



Review

- Find the derivative of $(3x - 2x^2)(5 + 4x)$

- $-24x^2 + 4x + 15$

- Find the derivative of $\frac{5x - 2}{x^2 + 1}$

- $\frac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$



4.3

The Chain Rule

Composition of Functions


- A composition of functions is simply putting 2 functions together—one inside the other.
- Example: In order to convert Fahrenheit to Kelvin we have to use a 2-step process by first converting Fahrenheit to Celsius.
$$C = (F - 32) \frac{5}{9}$$
$$K = C + 273$$
- $89^{\circ}\text{F} = 31.7^{\circ}\text{C}$
- $31.7^{\circ}\text{C} = 304.7\text{K}$
- But if we put 1 function inside the other function, then it is a 1-step process.



Composition of Functions

The composite of $f(x)$ and $g(x)$ is denoted $(f \circ g)(x)$ which means the same as $f(g(x))$.

- We are used to writing $f(x)$. $f(g(x))$ simply means that $g(x)$ is our new x in the f equation.
- We can also go the other way. $(g \circ f)(x)$ means $g(f(x))$.



Given $f(x) = 4x^2 - 2x$ $g(x) = 2x$

$$f(g(3)) = \qquad g(3) = 6$$

$$= f(6)$$

$$= 4(6)^2 - 2(6)$$

$$= 144 - 12$$

$$= 132$$

Given

$$f(x) = \frac{1}{x} \quad g(x) = x + 1$$

$$(f \circ g)(x) =$$

$$= f(g(x))$$

$$= f(x+1)$$

$$g(x) = x+1$$

$$= \frac{1}{x+1}$$

**Substitute $x+1$
In place of the
 x in the f equation**

$$(g \circ f)(x) =$$

$$= g(f(x))$$

$$= g\left(\frac{1}{x}\right)$$

$$=$$

$$= \frac{1}{x} + 1$$

**The new x in the g
equation**



The Chain Rule

CHAIN RULE

If y is a function of u , say $y = f(u)$, and if u is a function of x , say $u = g(x)$, then $y = f(u) = f[g(x)]$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Chain Rule Example

- Use the chain rule to find $D_x(x^2 + 5x)^8$
- Let $u = x^2 + 5x$
- Let $y = u^8$

$$\frac{dy}{dx} = \left(\frac{dy}{du} \right) \left(\frac{du}{dx} \right)$$

$$= 8u^7 (2x + 5)$$

$$= 8(x^2 + 5x)^7 (2x + 5)$$

Another way to think of it: The derivative of the outside times the derivative of the inside



Chain Rule – You Try

- Use the chain rule to find $D_x(3x - 2x^2)^3$
- Let $u = 3x - 2x^2$
- Let $y = u^3$

$$\frac{dy}{dx} = \left(\frac{dy}{du} \right) \left(\frac{du}{dx} \right)$$

The derivative of the
outside times the
derivative of the inside

$$= 3u^2 (3 - 4x)$$

$$= 3(3x - 2x^2)^2 (3 - 4x)$$



Chain Rule

- Find the derivative of $y = 4x(3x + 5)^5$
- This is the Product Rule inside the Chain Rule.
- Let $u = 3x + 5$; $y = u^5$

$$4x \left[5u^4(3) \right] + (3x + 5)^5(4)$$

$$4x \left[5(3x + 5)^4(3) \right] + 4(3x + 5)^5$$

$$4x \left(15(3x + 5)^4 \right) + 4(3x + 5)^5$$

$$60x(3x + 5)^4 + 4(3x + 5)^5$$



Chain Rule

$$= 60x(3x + 5)^4 + 4(3x + 5)^5$$

Factor out the common factor

$$= 4(3x + 5)^4 [15x + (3x + 5)]$$

$$= 4(3x + 5)^4 (18x + 5)$$



Chain Rule

- Find the derivative of $\frac{(3x+2)^7}{x-1}$
- This is the Quotient Rule in the Chain Rule
- Let $u = 3x + 2$; let $y = u^7$

$$= \frac{(x-1)[7u^6(3)] - (3x+2)^7(1)}{(x-1)^2}$$

$$= \frac{(x-1)[7(3x+2)^6(3)] - (3x+2)^7}{(x-1)^2}$$

$$= \frac{21[(x-1)(3x+2)^6] - (3x+2)^7}{(x-1)^2}$$



Chain Rule

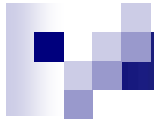
$$= \frac{21[(x-1)(3x+2)^6] - (3x+2)^7}{(x-1)^2}$$

Factor out the common factor

$$= \frac{(3x+2)^6 [21(x-1) - (3x+2)]}{(x-1)^2}$$

$$= \frac{(3x+2)^6 [21x - 21 - 3x - 2]}{(x-1)^2}$$

$$= \frac{(3x+2)^6 (18x - 23)}{(x-1)^2}$$



4.4

Derivatives of Exponential Functions



Derivative of e^x

DERIVATIVE OF e^x

$$\frac{d}{dx} (e^x) = e^x$$



Derivative of a^x

DERIVATIVE OF a^x

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

(The derivative of an exponential function is the original function times the natural logarithm of the base.)

$$D_x 3^x = (\ln 3)3^x$$



Other Derivatives

DERIVATIVE OF $a^{g(x)}$ AND $e^{g(x)}$

$$\frac{d}{dx} (a^{g(x)}) = (\ln a) a^{g(x)} g'(x)$$

and

$$\frac{d}{dx} (e^{g(x)}) = e^{g(x)} g'(x)$$



Examples – Find the Derivative

- $y = e^{5x}$

$$= e^{g(x)}(g'(x))$$

$$= e^{5x}(5) = 5e^{5x}$$



Examples – Find the Derivative

- $y = 3^{2x+1}$

$$= \ln a \left(a^{g(x)} \right) g'(x)$$

$$= \ln 3 \left(3^{2x+1} \right) (2)$$

$$= 2 \ln 3 \left(3^{2x+1} \right)$$



Example

- Find $\frac{dy}{dx}$ if $y = e^{x^2+1}\sqrt{5x+2}$
- Use the product rule

$$\begin{aligned}y &= e^{x^2+1} \left(D_x (5x+2)^{\frac{1}{2}} \right) + \sqrt{5x+2} \left(D_x \left(e^{x^2+1} \right) \right) \\&= \frac{1}{2} (5x+2)^{-\frac{1}{2}} (5) &= e^{x^2+1} (2x) \\&= \frac{5}{2\sqrt{5x+2}}\end{aligned}$$



Example

$$\begin{aligned}y &= e^{x^2+1} \left(D_x (5x+2)^{\frac{1}{2}} \right) + \sqrt{5x+2} \left(D_x \left(e^{x^2+1} \right) \right) \\&= e^{x^2+1} \frac{5}{2\sqrt{5x+2}} + \sqrt{5x+2} \left(2xe^{x^2+1} \right) \\&= \frac{5e^{x^2+1}}{2\sqrt{5x+2}} + \sqrt{5x+2} \left(2xe^{x^2+1} \right)\end{aligned}$$



Example Continued

$$= \frac{5e^{x^2+1}}{2\sqrt{5x+2}} + \left(\left(2xe^{x^2+1} \right) \sqrt{5x+2} \right) \left(\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} \right)$$

$$= \frac{5e^{x^2+1} + e^{x^2+1}(4x)(5x+2)}{2\sqrt{5x+2}}$$

$$= \frac{e^{x^2+1} (5 + 4x(5x+2))}{2\sqrt{5x+2}}$$

$$= \frac{e^{x^2+1} (20x^2 + 8x + 5)}{2\sqrt{5x+2}}$$

Get a common denominator to add the 2 parts together



4.5

Derivatives of Logarithmic Functions



Definition

DERIVATIVE OF $\log_a x$

$$\frac{d}{dx} [\log_a x] = \frac{1}{(\ln a)x}$$

(The derivative of a logarithmic function is the reciprocal of the product of the variable and the natural logarithm of the base.)

Bases – a side note

- Everything we do is in Base 10.

- We count up to 9, then start over. We change our numbering every 10 units.

1
2
3
4
5
6
7
8
9
10

Ones
Place

11
12
13
14
15
16
17
18
19
20

One
group of
ten and
1, 2,
3...ones

21
22
23...

Two
tens
and
...one
s

Bases

- The Yuki of Northern California used Base 8.
 - They counted up to 7, then started over. The numbering changed every 8 units.

1
2
3
4
5
6
7
10
11
12

Ones Place

13
14
15
16
17
20
21
22
23
24

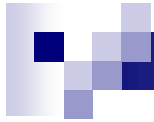
One eight and 3...ones

25
26
27...

Two eights and ...ones

So, 17 in Base 8 = 15 in Base 10

$$25_8 = 2 \text{ eights} + 5 \text{ ones} = 21$$



Bases

- The Mayans used Base 20.
- The Sumerians and people of Mesopotamia used Base 60.



Definition

DERIVATIVE OF $\ln x$

$$\frac{d}{dx} [\ln x] = \frac{1}{x}$$



Example

- Find $f'(x)$ if $f(x) = \ln 6x$
- Remember the properties of logs
- $\ln 6x = \ln 6 + \ln x$

$$\begin{aligned} & \frac{d}{dx}(\ln 6) + \frac{d}{dx}(\ln x) \\ &= 0 + \frac{1}{x} = \frac{1}{x} \end{aligned}$$



Definitions

DERIVATIVE OF $\log_a|x|$, $\log_a|g(x)|$, $\ln|x|$, AND $\ln|g(x)|$

$$\frac{d}{dx} [\log_a|x|] = \frac{1}{(\ln a)x}$$

$$\frac{d}{dx} [\log_a|g(x)|] = \frac{1}{\ln a} \cdot \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx} [\ln|x|] = \frac{1}{x}$$


$$\frac{d}{dx} [\ln|g(x)|] = \frac{g'(x)}{g(x)}$$



Examples – Find the Derivatives

- $y = \ln 5x$
- If $g(x) = 5x$, then $g'(x) = 5$

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{5}{5x} = \frac{1}{x}$$



$F'(x)$

- $f(x) = 3x \ln x^2$
- Product Rule

$$\begin{aligned} f'(x) &= (3x) \left[\frac{d}{dx} \ln x^2 \right] + (\ln x^2)(3) \\ &= 3x \left(\frac{2x}{x^2} \right) + (\ln x^2)(3) \\ &= 6 + 3 \ln x^2 \end{aligned}$$